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A SIMPLE MODEL OF TREES FOR UNICELLULAR MAPS

GUILLAUME CHAPUY, VALENTIN FÉRAY AND ÉRIC FUSY

Abstract. We consider unicellular maps, or polygon gluings, of fixed genus. A few years ago the first author gave a recursive bijection transforming unicellular maps into trees, explaining the presence of Catalan numbers in counting formulas for these objects. In this paper, we give another bijection that explicitly describes the "recursive part" of the first bijection. As a result we obtain a very simple description of unicellular maps as pairs made by a plane tree and a permutation-like structure.

All the previously known formulas follow as an immediate corollary or easy exercise, thus giving a bijective proof for each of them, in a unified way. For some of these formulas, this is the first bijective proof, *e.g.* the Harer-Zagier recurrence formula, the Lehman-Walsh formula and the Goupil-Schaeffer formula. We also discuss several applications of our construction: we obtain a new proof of an identity related to covered maps due to Bernardi and the first author, and thanks to previous work of the second author, we give a new expression for Stanley character polynomials, which evaluate irreducible characters of the symmetric group. Finally, we show that our techniques apply partially to unicellular 3-constellations and to related objects that we call quasi-3-constellations.

1. Introduction

A unicellular map is a connected graph embedded in a surface in such a way that the complement of the graph is a topological disk. These objects have appeared frequently in combinatorics in the last forty years, in relation with the general theory of map enumeration, but also with the representation theory of the symmetric group, the study of permutation factorizations, the computation of matrix integrals or the study of moduli spaces of curves. All these connections have turned the enumeration of unicellular maps into an important research field (for the many connections with other areas, see [22] and references therein; for an overview of the results see the introductions of the papers [12, 2]) The counting formulas for unicellular maps that appear in the literature can be roughly separated into two types.

The first type deals with *colored* maps (maps endowed with a mapping from its vertex set to a set of q colors). This implies "summation" enumeration formulas (see [19, 29, 24] or paragraph 3.4 below). These formulas are often elegant, and different combinatorial proofs for them have been given in the past few years [23, 17, 29, 24, 2]. The issue is that some important topological information, such as the genus of the surface, is not apparent in these constructions.

Key words and phrases. one-face map, Stanley character polynomial, bijection, Harer-Zagier formula, Rémy's bijection.

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Formulas of the second type keep track explicitly of the genus of the surface; they are either inductive relations, like the Harer-Zagier recurrence formula [19], or are explicit (but quite involved) closed forms, like the Lehman-Walsh [32] and the Goupil-Schaeffer [18] formulas. From a combinatorial point of view, these formulas are harder to understand. A step in this direction was done by the first author in [12] (this construction is explained in subsection 2.2), which led to new induction relations and to new formulas. However the link with other formulas of the second type remained mysterious, and [12] left open the problem of finding combinatorial proofs of these formulas.

The goal of this paper is to present a new bijection between unicellular maps and surprisingly simple objects which we call *C-decorated* trees (these are merely plane trees equipped with a certain kind of permutation on their vertices). This bijection, presented in Section 2, is based on the previous work of the first author [12]: we explicitly describe the "recursive part" appearing in this work. As a consequence, not only can we reprove all the aforementioned formulas in a bijective way, thus giving the first bijective proof for several of them, but we do that in a unified way. Indeed, C-decorated trees are so simple combinatorial objects that all formulas follow from our bijection as an immediate corollary or easy exercise, as we will see in Section 3.

Another interesting application of this bijection, studied in Section 4, is a new explicit way of computing the so-called Stanley character polynomials. The latter are nothing but the evaluation of irreducible characters of the symmetric groups, properly normalized and parametrized. Indeed, in a previous work [14], the second author expressed these polynomials as a generating function of (properly weighted) unicellular maps. Although we do not obtain a "closed form" expression (there is no reason to believe that such a form exists!), we express Stanley character polynomials as the result of a term-substitution in free cumulants, which are another meaningful quantity in representation theory of symmetric groups.

In Section 5 we discuss the possibility of applying our tools to *m-constellations*. This notion is a generalization of the notion bipartite maps introduced in connection with the study of factorizations in the symmetric group. A remarkable formula by Poulalhon and Schaeffer [26] (proved with the help of algebraic tools) suggests the possibility of a combinatorial proof using technique similar to ours. Although our bijection does not apply to these objects, we present two partial results in this direction, in the case of 3-constellations. One of them is an enumeration formula for a related family of objects that we call *quasi-3-constellations*, that turns out to be surprisingly similar to the Poulalhon-Schaeffer formula.

2. The main bijection

2.1. **Unicellular maps and C-decorated trees.** We first briefly review some standard terminology for maps.

A *map* M of genus $g \geq 0$ is a connected graph G embedded on a closed compact oriented[1] surface S of genus g, such that $S \backslash G$ is a collection of topological disks, which are called the *faces* of M. Loops and multiple edges are allowed. The

[1]Maps can also be defined on non-orientable surfaces. However, for non-orientable surfaces, only an asymptotic version [4] of the bijection of [12] has been discovered so far. Since the (complete) bijection of [12] is an essential building block of all the results presented in the present paper, we will not talk of non-orientable surfaces in this paper.

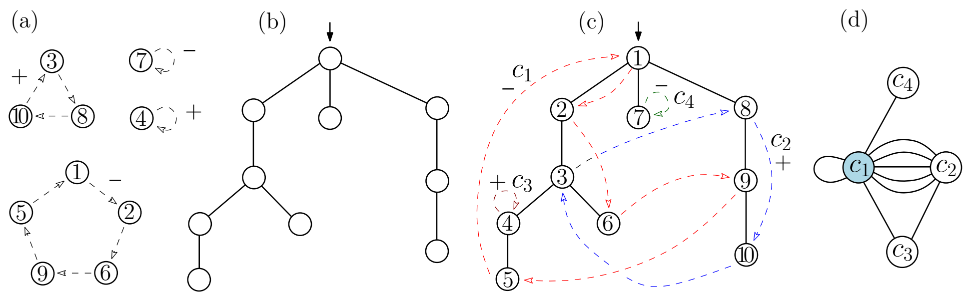


FIGURE 1. (a) A C-permutation σ. (b) A plane tree T. (c) The C-decorated tree (T, σ). (d) The underlying graph of (T, σ).

(multi)graph G is called the *underlying graph* of M and S its *underlying surface*. Two maps that differ only by an oriented homeomorphism between the underlying surfaces are considered the same. A corner of M is the angular sector between two consecutive edges around a vertex. A *rooted map* is a map with a marked corner, called the *root*; the vertex incident to the root is called the *root-vertex*. By convention, the map with one vertex and no edge (of genus 0) is considered as rooted at its unique vertex (the entire sector around the vertex is considered as a corner, which is the root).From now on, all maps are assumed to be rooted (note that the underlying graph of a rooted map is naturally vertex-rooted). A *unicellular map* is a map with a unique face. The classical Euler relation $|V| - |E| + |F| = 2 - 2g$ ensures that a unicellular map with n edges has $n + 1 - 2g$ vertices. A *plane tree* is a unicellular map of genus 0.

A *rotation system* on a connected graph G consists in a cyclic ordering of the half-edges of G around each vertex. Given a map M, its underlying graph G is naturally equipped with a rotation system given by the *clockwise ordering* of half-edges on the surface in a vicinity of each vertex. It is well-known that this correspondence is 1-to-1, i.e., a map can be considered as a connected graph equipped with a rotation system (thus, as a purely combinatorial object). We will take this viewpoint from now on.

We now introduce a new object called C-decorated tree.

A *cycle-signed* permutation is a permutation where each cycle carries a sign, either $+$ or $-$. A *C-permutation* is a cycle-signed permutation where all cycles have odd length, see Figure 1(a). For each C-permutation σ on n elements, the *rank* of σ is defined as $r(\sigma) = n - \ell(\sigma)$, where $\ell(\sigma)$ is the number of cycles of σ. Note that $r(\sigma)$ is even since all cycles have odd length. The *genus* of σ is defined as $r(\sigma)/2$. A *C-decorated tree* on n edges is a pair $\gamma = (T, \sigma)$ where T is a plane tree with n edges and σ is a C-permutation of $n + 1$ elements. The *genus* of γ is defined to be the genus of σ. Note that the $n + 1$ vertices of T can be canonically numbered from 1 to $n+1$ (*e.g.*, following a left-to-right depth-first traversal), hence σ can be seen as a permutation of the vertices of T, see Figure 1(c). The *underlying graph* of γ is the (vertex-rooted) graph G with n edges that is obtained from T by merging into a single vertex the vertices in each cycle of σ (so that the vertices of G correspond to the cycles of σ), see Figure 1(d).

Definition 1. *For n, g nonnegative integers, denote by $\mathcal{E}_g(n)$ the set of unicellular maps of genus g with n edges; and denote by $\mathcal{T}_g(n)$ the set of C-decorated trees of genus g with n edges.*

For two finite sets $\mathcal{A}$ and $\mathcal{B}$, we denote by $\mathcal{A}+\mathcal{B}$ their disjoint union and by $k\mathcal{A}$ the set made of k disjoint copies of $\mathcal{A}$. Besides, we write $\mathcal{A} \simeq \mathcal{B}$ if there is a bijection between $\mathcal{A}$ and $\mathcal{B}$. Our main result will be to show that $2^{n+1}\mathcal{E}_g(n) \simeq \mathcal{T}_g(n)$, with a bijection which preserves the underlying graphs of the objects.

2.2. **Recursive decomposition of unicellular maps.** In this section, we briefly recall a combinatorial method developed in [12] to decompose unicellular maps.

Proposition 1 (Chapuy [12])**.** *For $k \geq 1$, denote by $\mathcal{E}_g^{(2k+1)}(n)$ the set of maps from $\mathcal{E}_g(n)$ in which a set of $2k+1$ vertices is distinguished. Then for $g > 0$ and $n \geq 0$,*

$$2g\,\mathcal{E}_g(n) \simeq \mathcal{E}_{g-1}^{(3)}(n) + \mathcal{E}_{g-2}^{(5)}(n) + \mathcal{E}_{g-3}^{(7)}(n) + \cdots + \mathcal{E}_0^{(2g+1)}(n). \tag{1}$$

In addition, if M and (M', S') are in correspondence, then the underlying graph of M is obtained from the underlying graph of M' by merging the vertices in S' into a single vertex.

We now sketch briefly the construction of [12]. Although this is not really needed for the sequel, we believe that it gives a good insight into the objects we are dealing with (readers in a hurry may take Proposition 1 for granted and jump directly to subsection 2.3). We refer to [12] for proofs and details.

We first explain where the factor $2g$ comes from in (1). Let M be a rooted unicellular map of genus g with n edges. Then M has $2n$ corners, and we label them from 1 to $2n$ incrementally, starting from the root, and going clockwise around the (unique) face of M (Figure 2). Let v be a vertex of M, let k be its degree, and let $(a_1, a_2, \ldots, a_k)$ be the sequence of the labels of corners incident to it, read in *clockwise direction around* v starting from the minimal label $a_1 = \min\{a_i\}$. If for some j lying in $[\![1, k-1]\!]$ (*i.e.* in the set of integers between 1 and $k-1$, including 1 and $k-1$), we have $a_{j+1} < a_j$, we say that the corner of v labelled by a_{j+1} is a *trisection of* M. Figure 2(a) shows a map of genus two having four trisections. More generally we have:

Lemma 2 ([12])**.** *A unicellular map of genus g contains exactly $2g$ trisections. In other words, the set of unicellular maps of genus g with n edges and a marked trisection is isomorphic to $2g\,\mathcal{E}_g(n)$.*

Now, let τ be a trisection of M of label $a(\tau)$, and let v be the vertex it belongs to. We denote by c the corner of v with minimum label and by c' the corner with minimum label among those which appear between c and τ clockwise around v and whose label is greater than $a(\tau)$. By definition of a trisection, c' is well defined. We then construct a new map M', by *slicing* the vertex v into three new vertices using the three corners c, c', τ as in Figure 2(b). We say that the map M' is obtained from M by *slicing the trisection* τ. As shown in [12], the new map M' is a unicellular map of genus $g-1$. We can thus relabel the $2n$ corners of M' from 1 to $2n$, according to the procedure we already used for M. Among these corners, three of them, say c_1, c_2, c_3 are naturally inherited from the slicing of v, as on Figure 2(b). Let v_1, v_2, v_3 be the vertices they belong to, respectively. Then the following is true [12]: *In the map M', the corner c_i has the smallest label around the vertex v_i,*

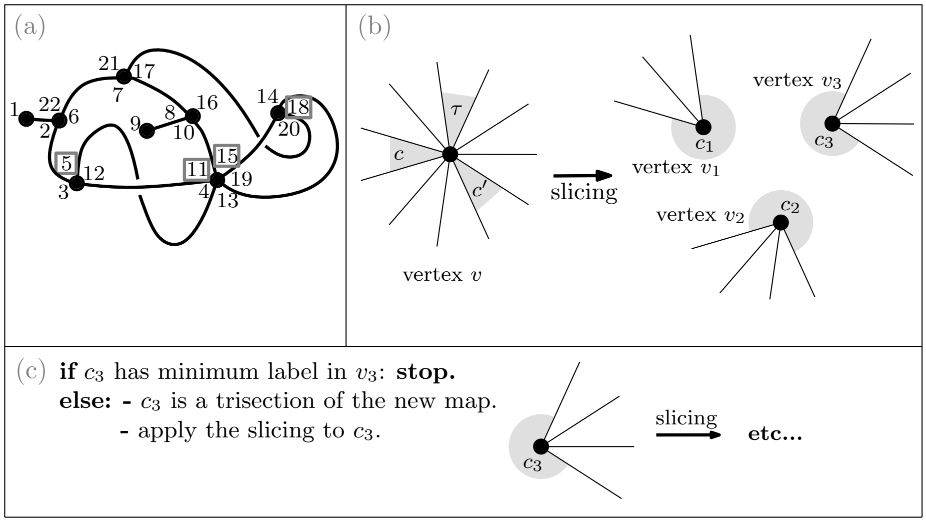


FIGURE 2. (a) A unicellular map of genus 2 equipped with its corner labelling. Labels corresponding to trisections are boxed. (b) Given a trisection τ, two other corners of interest c and c' are canonically defined (see text). "Slicing the trisection" then gives rise to three new vertices v_1, v_2, v_3, with distinguished corners c_1, c_2, c_3. (c) The recursive procedure of [12]: if c_3 is the minimum corner of v_3, then stop; else, as shown in [12], c_3 is a trisection of the new map M': in this case, iterate the slicing operation on (M', c_3).

for $i \in \{1,2\}$. For $i = 3$, either the same is true, or c_3 is a trisection of the map M'.

We now finally describe the bijection promised in Proposition 1. It is defined recursively on the genus, as follows. Given a map $M \in \mathcal{E}_g(n)$ with a marked trisection τ, let M' be obtained from M by slicing τ, and let c_i, v_i be defined as above for $i \in \{1,2,3\}$. If c_3 has the minimum label at v_3, set $\Psi(M,\tau) := (M', \{v_1, v_2, v_3\})$, which is an element of $\mathcal{E}^{(3)}_{g-1}(n)$. Otherwise, let $(M'', S) = \Psi(M', c_3)$, and set $\Psi(M,\tau) := (M'', S \cup \{v_1, v_2\})$. Note that this recursive algorithm necessarily stops, since the genus of the map decreases and since there are no trisections in unicellular maps of genus 0 (plane trees). Thus this procedure yields recursively a mapping that associates to a unicellular map M with a marked trisection τ another unicellular map M'' of a smaller genus, with a set S'' of marked vertices (namely the set of vertices which have been involved in a slicing at some point of the procedure). The set S'' of marked vertices necessarily has odd cardinality, as easily seen by induction. Moreover, it is clear that the underlying graph of M coincides with the underlying graph of M'' in which the vertices of S'' have been identified together into a single vertex. One can show that Ψ is a bijection [12], with an explicit inverse mapping.

2.3. **Recursive decomposition of C-decorated trees.** We now propose a recursive method to decompose C-decorated trees, which can be seen as parallel to

$\gamma = {}^{-}4\,7\,2\,3\,9\,6\,1\,5\,8$ $\qquad 4\,7|2\,3\,9\,6|1\,5\,8$ $\qquad 4\,7|2\,3\,9\,6|1\,5\,8$

$4\,7|2\,3\,9\,6|^{+}(1\,5\,8)$ $\qquad 4\,7|3|^{-}(2\,9\,6)|^{+}(1\,5\,8)$ $\qquad 4\,7|^{+}(3)|^{-}(2\,9\,6)|^{+}(1\,5\,8)$

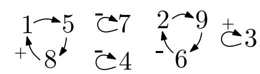
$7|^{-}(4)|^{+}(3)|^{-}(2\,9\,6)|^{+}(1\,5\,8)$ $\qquad {}^{-}(7)|^{-}(4)|^{+}(3)|^{-}(2\,9\,6)|^{+}(1\,5\,8)$ $\qquad \pi =$ 

FIGURE 3. The bijection between signed sequences and C-permutations.

the decomposition of unicellular maps given in the previous section. Denote by $\mathcal{C}(n)$ (resp. $\mathcal{C}_g(n)$) the set of C-permutations on n elements (resp. on n elements and of genus g). A *signed sequence* of integers is a pair (ϵ, S) where S is an integer sequence and ϵ is a sign, either $+$ or $-$. We will often write signed sequences with the sign preceding the sequence as a exponent, such as ${}^{\epsilon}S$.

Lemma 3. *Let X be a finite non-empty set of positive integers. Then there is a bijection between signed sequences of distinct integers from X —all elements of X being present in the sequence— and C-permutations on the set X. In addition the C-permutation has one cycle if and only if the signed sequence has odd length and starts with its minimal element.*

Proof. The bijection is illustrated in Figure 3. Starting from a signed sequence ${}^{\epsilon}S$, decompose S into blocks according to the left-to-right minimum records. Then treat the blocks successively from right to left. At each step, if the treated block B has odd length, turn B into the signed cycle ${}^{+}(B)$; if B has even length, move the second element x of B out of B, insert it at the end of the block preceeding B, and then turn B into the signed cycle ${}^{-}(B)$. Update the block-decomposition (according to left-to-right minimum records) on the left of B (it is very simple, two cases occur: if x is the minimum of the elements on the left of B, it occupies a single block; if not, x is integrated at the end of the block on the left of B). At the end of the right-to-left traversal, the last treated block B has odd length, we produce ${}^{\epsilon}(B)$ as the last signed cycle. The output is the C-permutation π made of all the signed cycles that have been produced during the traversal.

Conversely (read Figure 3 from right to left and bottom to top), starting from a C-permutation π, write π as the ordered list of its signed cycles, each cycle starting with its minimal element, and the cycles being ordered from left to right such that the minimal elements are in descending order. Record the sign ϵ_0 of the leftmost signed cycle ${}^{\epsilon_0}(B)$, and turn (B) into the block B. Then treat the signed cycles from left to right (starting with the second one). At each step, let ${}^{\epsilon}(B)$ be the treated signed cycle and let B' be the block to the left of ${}^{\epsilon}(B)$. Turn ${}^{\epsilon}(B)$ into the block B, and in case $\epsilon = -$, move the last element of B' to the second position of B (this possibly makes B' empty, in which case we erase B' of the current list of blocks). At the end, we get an ordered list of blocks, which can be seen as a sequence S. The output is the signed sequence ${}^{\epsilon_0}S$.

It is easy to see that the two mappings Φ (from signed sequences to C-permutations) and Ψ (from C-permutations to signed sequences) are inverse of each other; indeed these two mappings consist of a sequence of steps that operate on hybrid structures (a sequence of blocs followed by a sequence of signed cycles, these all start with their minimal element, and the minimal elements decrease from left to right), each step of Φ (resp. Ψ) increases (resp. decreasing) by 1 the number of signed cycles in the hybrid structure, and the step of Φ with i signed cycles is the inverse of the step of Ψ with $i+1$ signed cycles. $\square$

An element of a C-permutation is called *non-minimal* if it is not the minimum in its cycle. Non-minimal elements play the same role for C-permutations (and C-decorated trees) as trisections for unicellular maps. Indeed, a C-permutation of genus g has $2g$ non-minimal elements (compare with Lemma 2), and moreover we have the following analogue of Proposition 1:

Proposition 4. *For $k \geq 1$, denote by $\mathcal{T}_g^{(2k+1)}(n)$ the set of C-decorated trees from $\mathcal{T}_g(n)$ in which a set of $2k+1$ cycles is distinguished. Then for $g > 0$ and $n \geq 0$,*

$$2g\,\mathcal{T}_g(n) \simeq \mathcal{T}_{g-1}^{(3)} + \mathcal{T}_{g-2}^{(5)} + \mathcal{T}_{g-3}^{(7)} + \cdots + \mathcal{T}_0^{(2g+1)}.$$

In addition, if γ and (γ', S') are in correspondence, then the underlying graph of γ is obtained from the underlying graph of γ' by merging the vertices corresponding to cycles from S' into a single vertex.

Proof. For $k \geq 1$ let $\mathcal{C}_g^{(2k+1)}(n)$ be the set of C-permutations from $\mathcal{C}_g(n)$ where a subset of $2k+1$ cycles are marked. Let $\mathcal{C}_g^{\circ}(n)$ be the set of C-permutations from $\mathcal{C}_g(n)$ where a non-minimal element is marked. Note that $\mathcal{C}_g^{\circ}(n) \simeq 2g\,\mathcal{C}_g(n)$ since a C-permutation in $\mathcal{C}_g(n)$ has $2g$ non-minimal elements.

We now claim that $\mathcal{C}_g^{\circ}(n) \simeq \sum_{k=1}^{g} \mathcal{C}_{g-k}^{(2k+1)}(n)$. Indeed starting from $\gamma \in \mathcal{C}_g^{\circ}(n)$, write the signed cycle containing the marked element i of γ as a signed sequence beginning with i and apply Lemma 3 to this signed sequence: this produces a collection of $(2k+1) \geq 3$ signed cycles of odd length, which we take as the marked cycles.

We have thus shown that $2g\,\mathcal{C}_g(n) \simeq \sum_{k=1}^{g} \mathcal{C}_{g-k}^{(2k+1)}(n)$. Since by definition $\mathcal{T}_g(n) = \mathcal{E}_0(n) \times \mathcal{C}_g(n+1)$, we conclude that $2g\,\mathcal{T}_g(n) \simeq \sum_{k=1}^{g} \mathcal{T}_{g-k}^{(2k+1)}(n)$. The statement on the underlying graph just follows from the fact that the procedure in Lemma 3 merges the marked cycles into a unique cycle. $\square$

2.4. **The main result.**

Theorem 5. *For each non-negative integers n and g we have*

$$2^{n+1}\mathcal{E}_g(n) \simeq \mathcal{T}_g(n).$$

In addition the cycles of a C-decorated tree naturally correspond to the vertices of the associated unicellular map, in such a way that the respective underlying graphs are the same.

Proof. The proof is a simple induction on g, whereas n is fixed. The case $g = 0$ is obvious, as there are 2^{n+1} different C-permutations of size $(n+1)$ and genus 0, corresponding to the 2^{n+1} ways of giving signs to the identity permutation. Let $g > 0$. The induction hypothesis ensures that for each $g' < g$, $2^{n+1}\mathcal{E}_{g'}^{(2k+1)}(n) \simeq \mathcal{T}_{g'}^{(2k+1)}(n)$, where the underlying graphs (taking marked vertices vertices into account) of corresponding objects are the same. Hence, by Propositions 1 and 4, we have $2g\,2^{n+1}\mathcal{E}_g(n) \simeq 2g\,\mathcal{T}_g(n)$, where the underlying graphs of corresponding objects are the same. Finally, one can extract from this $2g$-to-$2g$ correspondence a 1-to-1 correspondence (think of extracting a perfect matching from a $2g$-regular bipartite graph, which is possible according to Hall's marriage theorem). And obviously the extracted 1-to-1 correspondence, which realizes $2^{n+1}\mathcal{E}_g(n) \simeq \mathcal{T}_g(n)$, also preserves the underlying graphs. $\square$

2.5. **A fractional, or stochastic, formulation.** Even if this does not hinder enumerative applications to be detailed in the next section, we do not know of an effective (polynomial-time) way to implement the bijection of Theorem 5; indeed the last step of the proof is to extract a perfect matching from a $2g$-regular bipartite graph whose size is exponential in n.

What can be done effectively (in time complexity $O(gn)$) is a *fractional* formulation of the bijection. For a finite set X, let $\mathbb{C}\langle X\rangle$ be the set of linear combinations of the form $\sum_{x\in X} u_x \cdot x$, where the $x \in X$ are seen as independent formal vectors, and the coefficients u_x are in $\mathbb{C}$. Let $\mathbb{R}_1^+\langle X\rangle \subset \mathbb{C}\langle X\rangle$ be the subset of linear combinations where the coefficients are nonnegative and add up to 1. Denote by $\mathbf{1}_X$ the vector $\sum_{x\in X} x$. For two finite sets X and Y, a *fractional mapping* from X to Y is a linear mapping φ from $\mathbb{C}\langle X\rangle$ to $\mathbb{C}\langle Y\rangle$ such that the image of each $x \in X$ is in $\mathbb{R}_1^+\langle Y\rangle$; the set of elements of Y whose coefficients in $\varphi(x)$ are strictly positive is called the *image-support* of x. Note that $\varphi(x)$ identifies to a probability distribution on Y; a "call to $\varphi(x)$" is meant as picking up $y \in Y$ under this distribution. A fractional mapping is *bijective* if $\mathbf{1}_X$ is mapped to $\mathbf{1}_Y$, and is *deterministic* if each $x \in X$ is mapped to some $y \in Y$. Note that, if there is a fractional bijection from X to Y, then $|X| = |Y|$ (indeed in that case the matrix of φ is bistochastic).

One can now formulate by induction on the genus an effective (the cost of a call is $O(gn)$) fractional bijection from $2^{n+1}\mathcal{E}_g[n]$ to $\mathcal{T}_g(n)$, and similarly from $\mathcal{T}_g[n]$ to $2^{n+1}\mathcal{E}_g(n)$. The crucial property is that, for $k \geq 1$ and E, F finite sets, if there is a fractional bijection Φ from kE to kF then one can *effectively* derive from it a fractional bijection $\widetilde{\Phi}$ from E to F: for $x \in E$, just define $\widetilde{\Phi}(x)$ as $\frac{1}{k}(\iota(\Phi(x_1)) + \cdots + \iota(\Phi(x_k)))$, where $x_1, \ldots, x_k$ are the representatives of x in kE, and where ι is the projection from kF to F. In other words a call to $\widetilde{\Phi}(x)$ consists in picking up a representative x_i of x in kE uniformly at random and then calling $\Phi(x)$. Hence by induction on g, Propositions 1 and 4 (where the stated combinatorial isomorphisms are effective) ensure that there is an effective fractional bijection from $2^{n+1}\mathcal{E}_g(n)$ to $\mathcal{T}_g[n]$ and similarly from $\mathcal{T}_g[n]$ to $2^{n+1}\mathcal{E}_g[n]$, such that if γ' is in the image-support of γ then the underlying graphs of γ and γ' are the same.

Note that, given an effective fractional bijection between two sets X and Y, and a uniform random sampling algorithm on the set X, one obtains immediately a uniform random sampling algorithm for the set Y. In the next section, we will use our bijection to prove several enumerative formulas for unicellular maps, starting from elementary results on the enumeration of trees or permutations. In all cases, we will also be granted with a uniform random sampling algorithm for the corresponding unicellular maps, though we will not emphasize this point in the rest of the paper.

3. Counting formulas for unicellular maps

It is quite clear that C-decorated trees are much simpler combinatorial objects than unicellular maps. In this section, we use them to give bijective proofs of several known formulas concerning unicellular maps. We focus on the Lehman-Walsh and the Goupil-Schaeffer formulas, and the Harer-Zagier recurrence, of which bijective proofs were long-awaited. We also give new bijective proofs of several summation formulas (for which different bijective proofs are already known): the Harer-Zagier summation formula and a refinement of it, a formula due to Jackson for bipartite maps and its refinement due to A. Morales and E. Vassilieva. We finally consider an identity involving *covered maps*, obtained originally from a difficult bijection by the first author and O. Bernardi and which can be explained easily thanks to our new bijection. We insist on the fact that all these proofs are elementary consequences of our main bijection (Theorem 5).

3.1. **Two immediate corollaries.** The set $\mathcal{T}_g(n) = \mathcal{E}_0(n)\times\mathcal{C}_g(n+1)$ is the product of two sets that are easy to count. Precisely, let $\epsilon_g(n) = |\mathcal{E}_g(n)|$ and $c_g(n) = |\mathcal{C}_g(n)|$. Recall that $\epsilon_0(n) = \mathrm{Cat}(n)$, where $\mathrm{Cat}(n) := \frac{(2n)!}{n!(n+1)!}$ is the n-th Catalan number. Therefore Theorem 5 gives $\epsilon_g(n) = 2^{-n-1}\mathrm{Cat}(n)c_g(n+1)$.

One gets easily a closed form for $c_g(n+1)$ (by summing over all possible cycle types) and an explicit formula for the generating series, thereby recovering two classical results for the enumeration of unicellular maps.

Every partition of $n+1$ in $n+1-2g$ odd parts writes as $1^{n+1-2g-\ell}3^{m_1}\dots(2k+1)^{m_k}$ for some partition $\gamma = (\gamma_1,\dots,\gamma_\ell) = 1^{m_1}\dots k^{m_k}$ of g. The number $a_\gamma(n+1)$ of permutations of $n+1$ elements with cycle-type equal to $1^{n+1-2g-\ell}3^{m_1}\dots(2k+1)^{m_k}$ is classically given by

$$a_\gamma(n+1) = \frac{(n+1)!}{(n+1-2g-\ell)!\prod_i m_i!(2i+1)^{m_i}},$$

and the number of C-permutations with this cycle-type is just $a_\gamma(n+1)2^{n+1-2g}$ (since each cycle has 2 possible signs). Hence, we get the equality

$$c_g(n+1) = 2^{n+1-2g}\sum_{\gamma\vdash g} a_\gamma(n+1).$$

We thus recover:

Proposition 6 (Walsh and Lehman [32])**.** *The number $\epsilon_g(n)$ is given by*

$$\epsilon_g(n) = \frac{(2n)!}{n!(n+1-2g)!2^{2g}}\sum_{\gamma\vdash g}\frac{(n+1-2g)_\ell}{\prod_i m_i!(2i+1)^{m_i}},$$

where $(x)_k = \prod_{j=0}^{k-1}(x-j)$, ℓ is the number of parts of γ, and m_i is the number of parts of length i in γ.

Define the exponential generating function

$$C(x,y) := \sum_{n,g}\tfrac{1}{(n+1)!}c_g(n+1)y^{n+1}x^{n+1-2g}$$

of C-permutations where y marks the number of elements, which are labelled, and x marks the number of cycles. Since a C-permutation is a set of signed cycles of odd lengths, $C(x,y)$ is given by

$$C(x,y) = \exp\left(2x\sum_{k\geq 0}\frac{y^{2k+1}}{2k+1}\right) - 1.$$

Indeed the sum in the parenthesis is the generating function of cycles of odd lengths, the factor 2 is for the signs of cycles, the exp means that we take a set of such signed cycles, and the -1 means that the set is non-empty (see *e.g.* [15, Part A] for a general presentation of the methodology to translate classical combinatorial set operations into generating function expressions, in particular page 120 for the application to permutations seen as sets of cycles). Since $\sum_{k\geq 0} \frac{y^{2k+1}}{2k+1} = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right)$, the expression simplifies to

$$C(x,y) = \exp\left(x \log\left(\frac{1+y}{1-y}\right)\right) - 1 = \left(\frac{1+y}{1-y}\right)^x - 1.$$

Since $c_0(1) = 2$ and $\frac{1}{(n+1)!}c_g(n+1) = \frac{2^{n+1}n!}{(2n)!}\epsilon_g(n) = \frac{2}{(2n-1)!!}\epsilon_g(n)$ for $n \geq 1$, we recover:

Proposition 7 (Harer-Zagier series formula [19, 22])**.** *The generating function* $E(x,y) := 1 + 2xy + 2\sum_{g\geq 0, n>0} \frac{\epsilon_g(n)}{(2n-1)!!} y^{n+1}x^{n+1-2g}$ *is given by*

$$E(x,y) = \left(\frac{1+y}{1-y}\right)^x.$$

3.2. **Harer-Zagier recurrence formula.** Elementary algebraic manipulations on the expression of $E(x,y)$ yield a very simple recurrence satisfied by $\epsilon_g(n)$, known as the Harer-Zagier recurrence formula (stated in Proposition 10 hereafter). We now show that the model of C-decorated trees makes it possible to derive this recurrence directly from a combinatorial isomorphism, that generalizes Rémy's beautiful bijection [28] formulated on plane trees.

It is convenient here to consider C-decorated trees as *unlabelled structures*: precisely we see a C-decorated tree as a plane tree where the vertices are partitioned into parts of odd size, where each part carries a sign + or −, and such that the vertices in each part are cyclically ordered (the C-permutation can be recovered by numbering the vertices of the tree according to a left-to-right depth-first traversal), think of Figure 1(c) where the labels have been taken out. We take here the convention that a plane tree with n edges has $2n+1$ corners, considering that the sector of the root has two corners, one on each side of the root.

We denote by $\mathcal{P}(n) = \mathcal{E}_0(n)$ the set of plane trees with n edges, and by $\mathcal{P}^{\mathrm{v}}(n)$ (resp. $\mathcal{P}^{\mathrm{c}}(n)$) the set of plane trees with n edges where a vertex (resp. a corner) is marked. Rémy's procedure, shown in Figure 4, realizes the isomorphism $\mathcal{P}^{\mathrm{v}}(n) \simeq 2\,\mathcal{P}^{\mathrm{c}}(n-1)$, or equivalently

$$(n+1)\mathcal{P}(n) \simeq 2(2n-1)\mathcal{P}(n-1). \tag{2}$$

Let $\mathcal{T}_g^{\mathrm{v}}(n)$ be the set of C-decorated trees from $\mathcal{T}_g(n)$ where a vertex is marked. Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the subset of objects in $\mathcal{T}_g^{\mathrm{v}}(n)$ where the signed cycle containing the marked vertex has length 1 (resp. length greater than 1). Let $\gamma \in \mathcal{T}_g^{\mathrm{v}}(n)$, with $n \geq 1$. If $\gamma \in \mathcal{A}$, record the sign of the 1-cycle containing v and then apply Rémy's procedure to the plane tree with respect to v (so as to delete v). This reduction, which does not change the genus, yields $\mathcal{A} \simeq 2 \cdot 2(2n-1)\mathcal{T}_g(n-1)$. If $\gamma \in \mathcal{B}$, let c be the cycle containing the marked vertex v; c is of the form $(v, v_1, v_2, \ldots, v_{2k})$ for some $k \geq 1$. Move v_1 and v_2 out of c (the successor of v becomes the former successor of v_2). Then apply Rémy's procedure twice, firstly with respect to v_1 (on a plane tree with n edges), secondly with respect to v_2 (on a plane tree with $n-1$ edges). This reduction, which decreases the genus by 1, yields $\mathcal{B} \simeq 2(2n-1)2(2n-3)\mathcal{T}_{g-1}^{\mathrm{v}}(n-2)$, hence $\mathcal{B} \simeq 4(n-1)(2n-1)(2n-3)\mathcal{T}_{g-1}(n-2)$.

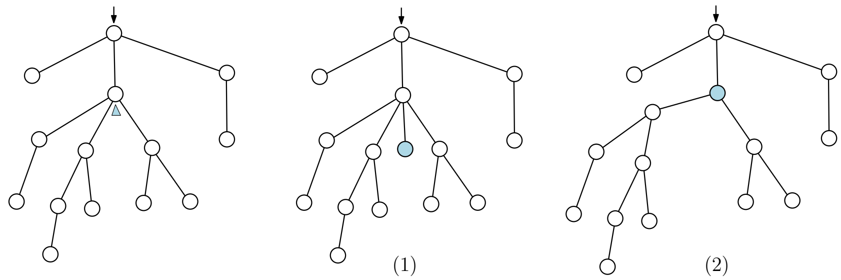


FIGURE 4. Rémy's procedure gives two ways to obtain a plane tree with n edges and a marked vertex v from a plane tree with $n-1$ edges and a marked corner: (1) in the first way (replacing the marked corner by a leg) v is a leaf, (2) in the second way (stretching an edge to carry the subtree on the left of the marked corner) v is a non-leaf.

Since $\mathcal{T}_g^{\mathrm{v}}(n) = \mathcal{A} + \mathcal{B}$ and $\mathcal{T}_g^{\mathrm{v}}(n) \simeq (n+1)\mathcal{T}_g(n)$, we finally obtain the isomorphism

$$(3) \quad (n+1)\mathcal{T}_g(n) \simeq 4(2n-1)\mathcal{T}_g(n-1) + 4(n-1)(2n-1)(2n-3)\mathcal{T}_{g-1}(n-2),$$

which holds for any $n \geq 1$ and $g \geq 0$ (with the convention $\mathcal{T}_g(n) = \emptyset$ if g or n is negative). Since $2^{n+1}\mathcal{E}_g(n) \simeq \mathcal{T}_g(n)$, we recover:

Proposition 8 (Harer-Zagier recurrence formula [19, 22])**.** *The coefficients $\epsilon_g(n)$ satisfy the following recurrence relation valid for any $g \geq 0$ and $n \geq 1$ (with $\epsilon_0(0) = 1$ and $\epsilon_g(n) = 0$ if $g < 0$ or $n < 0$):*

$$(n+1)\epsilon_g(n) = 2(2n-1)\epsilon_g(n-1) + (n-1)(2n-1)(2n-3)\epsilon_{g-1}(n-2).$$

To the best of our knowledge this is the first proof of the Harer-Zagier recurrence formula that directly follows from a combinatorial isomorphism. The isomorphism (3) also provides a natural extension to arbitrary genus of Rémy's isomorphism (2).

3.3. **Refined enumeration of bipartite unicellular maps.** In this paragraph, we explain how to recover a formula due to A. Goupil and G. Schaeffer [18, Theorem 2.1] from our bijection. Let us first give a few definitions. A graph is *bipartite* if its vertices can be colored in black and white such that each edge connects a black and a white vertex. If the graph has a root-vertex v, then v is required to be black; thus, if the graph is also connected, then such a bicoloration of the vertices is unique. From now on, a connected bipartite graph with a root-vertex is assumed to be endowed with this canonical bicoloration.

The degree distribution of a map/graph is the sequence of the degrees of its vertices taken in decreasing order (it is a partition of $2n$, where n is the number of edges). If we consider a bipartite map/graph, we can consider separately the *white vertex degree distribution* and the *black vertex degree distribution*, which are two partitions of n.

Let ℓ, m, n be positive integers such that $n+1-\ell-m$ is even. Fix two partitions λ, μ of n of respective lengths ℓ and m. We call $\mathrm{Bi}(\lambda, \mu)$ the number of bipartite unicellular maps, with white (resp. black) vertex degree distribution λ (resp. μ). The corresponding genus is $g = (n + 1 - \ell - m)/2$.

The purpose of this paragraph is to compute $\mathrm{Bi}(\lambda, \mu)$. It will be convenient to change a little bit the formulation of the problem and to consider *labelled maps* instead of the usual non-labelled maps: a *labelled map* is a map whose vertices are labelled with integers $1, 2, \cdots$. If the map is bipartite, we require instead that the white and black vertices are labelled separately (with respective labels $w_1, w_2, \cdots$ and $b_1, b_2, \cdots$). The degree distribution(s) of a labelled map (resp. bipartite labelled map) with n edges can be seen as a composition of $2n$ (resp. two compositions of n). For $\boldsymbol{I} = (i_1, \cdots, i_\ell)$ and $\boldsymbol{J} = (j_1, \cdots, j_m)$ two compositions of n, we denote by $\mathrm{BiL}(\boldsymbol{I}, \boldsymbol{J})$ the number of labelled bipartite unicellular maps with white (resp. black) vertex degree distribution $\boldsymbol{I}$ (resp. $\boldsymbol{J}$). The link between $\mathrm{Bi}(\lambda, \mu)$ and $\mathrm{BiL}(\boldsymbol{I}, \boldsymbol{J})$ is straightforward: $\mathrm{BiL}(\boldsymbol{I}, \boldsymbol{J}) = m_1(\lambda)! m_2(\lambda)! \cdots m_1(\mu)! m_2(\mu)! \cdots \mathrm{Bi}(\lambda, \mu)$, where λ and μ are the sorted versions of $\boldsymbol{I}$ and $\boldsymbol{J}$. We now recover the following formula:

Proposition 9 (Goupil and Schaeffer [18, Theorem 2.1])**.**

$$\mathrm{BiL}(\boldsymbol{I}, \boldsymbol{J}) = 2^{-2g} \cdot n \cdot \sum_{g_1 + g_2 = g} (\ell + 2g_1 - 1)!(m + 2g_2 - 1)! \sum_{\substack{p_1 + \cdots + p_\ell = g_1 \\ q_1 + \cdots + q_m = g_2}} \prod_{r=1}^{\ell} \frac{1}{2p_r + 1} \binom{i_r - 1}{2p_r} \prod_{r=1}^{m} \frac{1}{2q_r + 1} \binom{j_r - 1}{2q_r}. \tag{4}$$

Proof. For $g = 0$ the formula is simply

$$\mathrm{BiL}(\boldsymbol{I}, \boldsymbol{J}) = n(\ell - 1)!(m - 1)!, \tag{5}$$

which can easily be established by a bivariate version of the cycle lemma, see also [16, Theorem 2.2]. (Note that, in that case, the cardinality only depends on the lengths of $\boldsymbol{I}$ and $\boldsymbol{J}$.)

We now prove the formula for arbitrary g. Consider some lists $\boldsymbol{p} = (p_1, \cdots, p_\ell)$ and $\boldsymbol{q} = (q_1, \cdots, q_m)$ of nonnegative integers with total sum g: let $g_1 = \sum p_i$ and $g_2 = \sum q_i$. We say that a composition $\boldsymbol{H}$ *refines* $\boldsymbol{I}$ along $\boldsymbol{p}$ if $\boldsymbol{H}$ is of the form $(h_1^1, \cdots, h_1^{2p_1+1}, \cdots, h_\ell^1, \cdots, h_\ell^{2p_\ell+1})$, with $\sum_{t=1}^{2p_r+1} h_r^t = i_r$ for all r between 1 and ℓ. Clearly, there are $\prod_{r=1}^{\ell} \binom{i_r - 1}{2p_r}$ such compositions $\boldsymbol{H}$. One defines similarly a composition $\boldsymbol{K}$ refining $\boldsymbol{J}$ along $\boldsymbol{q}$.

Consider now the set of labelled bipartite plane trees of vertex degree distributions $\boldsymbol{H}$ and $\boldsymbol{K}$, where $\boldsymbol{H}$ (resp. $\boldsymbol{K}$) refines $\boldsymbol{I}$ (resp. $\boldsymbol{J}$) along $\boldsymbol{p}$ (resp. $\boldsymbol{q}$). By (5), there are $n \cdot (\ell + 2g_1 - 1)!(m + 2g_2 - 1)!$ trees for each pair $(\boldsymbol{H}, \boldsymbol{K})$, so in total, with $\boldsymbol{I}$, $\boldsymbol{J}$, $\boldsymbol{p}$ and $\boldsymbol{q}$ fixed, the number of such trees is:

$$n \cdot (\ell + 2g_1 - 1)!(m + 2g_2 - 1)! \prod_{r=1}^{\ell} \binom{i_r - 1}{2p_r} \prod_{r=1}^{m} \binom{j_r - 1}{2q_r}. \tag{6}$$

As the parts of $\boldsymbol{H}$ (resp. $\boldsymbol{K}$) are naturally indexed by pairs of integers, we can see these trees as labelled by the set $\{w_r^t; 1 \leq r \leq \ell, 1 \leq t \leq 2p_r + 1\} \sqcup \{b_r^t; 1 \leq r \leq m, 1 \leq t \leq 2q_r + 1\}$. There is a canonical permutation of the vertices of the trees with cycles of odd sizes and which preserves the bicoloration: just send w_r^t to w_r^{t+1} (resp. b_r^t to b_r^{t+1}), where $t + 1$ is meant modulo $2p_r + 1$ (resp. $2q_r + 1$). If we additionally put a sign on each cycle, we get a C-decorated tree (with labelled cycles) that corresponds to a labelled bipartite map with white (resp. black) vertex degree distribution $\boldsymbol{I}$ (resp. $\boldsymbol{J}$). Conversely, to recover a labelled bipartite plane tree from such a C-decorated tree, one has to choose in each cycle which vertex gets the label w_r^1 or b_r^1, and one has to forget the signs of the $(n+1-g)$ cycles. This represents a factor $2^{n+1-2g}\left(\prod_{r=1}^{\ell}(2p_r+1)\prod_{r=1}^{m}(2q_r+1)\right)^{-1}$.

Multiplying (6) by the above factor, and summing over all possible sequences $\boldsymbol{p}$ and $\boldsymbol{q}$ of total sum g, we conclude that the number of C-decorated trees associated with labelled bipartite unicellular maps of white (resp. black) vertex degree distribution $\boldsymbol{I}$ (resp. $\boldsymbol{J}$), is equal to 2^{n+1} times the right-hand side of (4). By Theorem 5, this number is also equal to $2^{n+1}\mathrm{BiL}(\boldsymbol{I},\boldsymbol{J})$. This ends the proof of Proposition 9. $\square$

This is the first combinatorial proof of (4) (the proof by Goupil and Schaeffer involves representation theory of the symmetric group). Moreover, the authors of [18] found surprising that "the two partitions contribute independently to the genus". With our approach, this is very natural, since the cycles are carried independently by white and black vertices.

Remark 1. If we set $\boldsymbol{J}=(2^m)$, we find the number of monochromatic maps with m edges and vertex-degree distribution $\boldsymbol{I}$. This explains why we did not consider separately the monochromatic and bipartite cases (as we do in the next section).

3.4. **Counting colored maps.** In this paragraph, we deal with what was presented in the introduction as the *first type* of formulas. These formulas give an expression for a certain *sum* of coefficients counting unicellular maps, the expressions being usually simpler than those for the counting coefficients taken separately (like the Goupil-Schaeffer's formula). These sums can typically be seen as counting formulas for *colored unicellular maps* (where the control is on the number of colors, which gives indirect access to the genus).

3.4.1. *A summation formula for unicellular maps.* We begin with Harer-Zagier's summation formula [19, 22] (which can also be very easily derived from the expression of $E(x,y)$). In contrast to the formulas presented so far, this one has already been given combinatorial proofs [23, 17, 2] using different bijective constructions, but we want to insist on the fact that our construction gives bijective proofs for all the formulas in a unified way.

Proposition 10 (Harer-Zagier summation formula [19, 22])**.** *Let $A(v;n)$ be the number of unicellular maps with n edges and v vertices. Then for $n\geq 1$*

$$\sum_v A(v;n)x^v=(2n-1)!!\sum_{r\geq 1}2^{r-1}\binom{n}{r-1}\binom{x}{r}.$$

Proof. To be comprehensive, let us begin by explaining the well-known combinatorial reformulation of this formula. Let $A_r(n)$ be the number of unicellular maps with n edges, each vertex having a color in $[\![1,r]\!]$, and each color in $[\![1,r]\!]$ being used at least once. Then

$$\sum_v A(v;n)x^v=\sum_{r\geq 1}A_r(n)\binom{x}{r}. \tag{7}$$

Indeed, both sides count the number of pairs (M,φ), where M is a unicellular map with n edges and φ is a mapping from the vertex set of M to a given set X of size x. For the left-hand side, this is clear: for a given map, there are x^v such mappings, where v is the number of vertices of the map. But we can count these pairs in another way. Let us consider the pairs (M, φ) for which the image set of φ is a given set $X' \subseteq X$. If r is the size of X', such a pair is the same thing as a unicellular map colored with colors in $[\![1, r]\!]$ and each color being used at least once. Therefore, for a fixed X', one has $A_r(n)$ such pairs (M, φ). As there are $\binom{x}{r}$ sets $X' \subseteq X$ of size r, there are in total

$$\sum_{r\geq 1} A_r(n)\binom{x}{r}$$

pairs (M, φ), which proves identity (7).

Thus, it suffices to prove that $A_r(n) = (2n-1)!!\,2^{r-1}\binom{n}{r-1}$. Our main bijection sends unicellular maps colored with colors in $[\![1, r]\!]$ (each color being used at least once) onto C-decorated trees with n edges, where each (signed) cycle has a color in $[\![1, r]\!]$, and such that each color in $[\![1, r]\!]$ is used by at least one cycle. Each of the r colors yields a (non-empty) C-permutation, which can be represented as a signed sequence, according to Lemma 3. Then one can encode these r signed sequences ${}^{\epsilon_1}S_1, {}^{\epsilon_2}S_2 \ldots, {}^{\epsilon_r}S_r$ by the triple (S, T, U) where $S = S_1S_2\ldots S_r$ is their concatenation (it is a sequence of length $n+1$), where $T = (\epsilon_1, \epsilon_2, \ldots, \epsilon_r)$ is the r-tuple giving their signs and where U is the subset of $r-1$ elements among the elements between positions 2 and $(n+1)$ in S, that indicates the starting elements of the sequences $S_2, \ldots, S_r$. For instance if $r = 3$ and if the signed sequences corresponding respectively to colors $1, 2, 3$ are $S_1 =^+ (3, 9, 4)$, $S_2 =^- (5, 8, 6, 2)$, and $S_3 =^- (1, 7)$, then the concatenated sequence is $S = (3, 9, 4, 5, 8, 6, 2, 1, 7)$, together with the 3 signs $T = (+, -, -)$ and the two selected elements $U = \{5, 1\}$. It is clear that this correspondence is bijective. Hence the number of such C-decorated trees is $(n+1)!\,2^r\binom{n}{r-1}$, and by Theorem 5,

$$A_r(n) = 2^{-n-1}\mathrm{Cat}(n)(n+1)!\,2^r\binom{n}{r-1} = (2n-1)!!\,2^{r-1}\binom{n}{r-1}. \qquad \square$$

3.4.2. *A summation formula for bipartite unicellular maps.* By Theorem 5, a C-decorated tree associated to a bipartite unicellular map is a bipartite plane tree such that each signed cycle must contain only white (resp. black) vertices. Recall that the $n+1$ vertices carry distinct labels from 1 to $n+1$ (the ordering follows by convention a left-to-right depth-first traversal, see Figure 1(c)). Without loss of information the i black vertices (resp. j white vertices) can be relabelled from 1 to i (resp. from 1 to j) in the order-preserving way; we take here this convention for labelling the vertices of such a C-decorated tree. We now recover the following summation formula due to Jackson (different bijective proofs have been given in [29] and in [2]):

Proposition 11 (Jackson's summation formula [21])**.** *Let $B(v, w; n)$ be the number of bipartite unicellular maps with n edges, v black vertices and w white vertices. Then for $n \geq 1$*

$$\sum_{v,w} B(v, w; n)y^v z^w = n! \sum_{r,s\geq 1}\binom{n-1}{r-1, s-1}\binom{y}{r}\binom{z}{s}.$$

Proof. As for the Harer-Zagier formula, there is a well-known combinatorial reformulation of this statement. Namely, it suffices to prove that, for $r, s \geq 1$, the number $B_{r,s}(n)$ of bipartite unicellular maps with n edges, each black (resp. white) vertex having a so-called *b-color* in $[\![1, r]\!]$ (resp. a so-called *w-color* in $[\![1, s]\!]$), such that each b-color in $[\![1, r]\!]$ (resp. w-color in $[\![1, s]\!]$) is used at least once, is given by $B_{r,s}(n) = n!\binom{n-1}{r-1, s-1}$. For n, i, j such that $i + j = n + 1$, consider a bipartite C-decorated tree with n edges, i black vertices, j white vertices, where each black (resp. white) signed cycle has a b-color in $[\![1, r]\!]$ (resp. a w-color in $[\![1, s]\!]$), and each b-color in $[\![1, r]\!]$ (resp. w-color in $[\![1, s]\!]$) is used at least once. By the same argument as in Proposition 10, the C-permutation and b-colors on black vertices can be encoded by a sequence S_b of length i of distinct integers in $[\![1, i]\!]$, together with a sequence of r signs and a subset of $r-1$ elements among the $i-1$ elements at positions from 2 to i in S_b. And the C-permutation and w-colors on white vertices can be encoded by a sequence S_w of length j of distinct integers in $[\![1, j]\!]$, together with a sequence of s signs and a subset of $s-1$ elements among the $j-1$ elements at positions from 2 to j in S_w. Hence there are $\mathrm{Nar}(i, j; n)2^{r+s}i!\binom{i-1}{r-1}j!\binom{j-1}{s-1}$ such C-decorated trees, where $\mathrm{Nar}(i, j; n)$ (called the *Narayana* number) is the number of bipartite plane trees with n edges, i black vertices and j white vertices, given by $\mathrm{Nar}(i, j; n) = \frac{1}{n}\binom{n}{i}\binom{n}{j}$. By Theorem 5,

$$\begin{aligned} B_{r,s}(n) = \quad & 2^{-n-1}2^{r+s}\sum_{i+j=n+1}\mathrm{Nar}(i, j; n)i!j!\binom{i-1}{r-1}\binom{j-1}{s-1} \\ = \quad & n!(n-1)!\frac{2^{r+s-n-1}}{(r-1)!(s-1)!}\sum_{\substack{i+j=n+1\\ i\geq r, j\geq s}}\frac{1}{(i-r)!(j-s)!}. \end{aligned}$$

But we have

$$\sum_{i+j=n+1}\frac{1}{(i-r)!(j-s)!} = \sum_{i+j=n+1-r-s}\frac{1}{i!j!} = \frac{2^{n+1-r-s}}{(n+1-r-s)!}.$$

Hence $B_{r,s}(n) = n!\binom{n-1}{r-1, s-1}$. ☐

3.4.3. *A refinement of the Harer-Zagier summation formula.* The proof method above can be used to keep track of the vertex degree distribution in the Harer-Zagier formula. Before stating the resulting formula, let us mention that other methods can also keep track of this statistics, for instance the bijective approach developed in [2][2]. Thus, although the formula had not yet been stated explicitly in the literature (as far as we know), all the elements needed to prove it were already there[3]. Of course, the proof presented here is new and fits in our unified framework.

Proposition 12. *Let* m_ρ *and* p_λ *be the monomial and power sum bases of the ring of symmetric functions and* $\boldsymbol{x}$ *be an infinite set of variables. We denote by* $\mathrm{Ai}(\lambda)$

[2] Proposition 12 can also be deduced from a formula of A. Morales and E. Vassilieva (Proposition 14 below) using [5, Lemma 9].

[3] This formula was known to an anonymous referee, who suggested we include its proof in the present paper.

the number of unicellular maps with degree distribution λ. Then, for any integer n, one has:

$$\sum_{\lambda \vdash 2n} \mathrm{Ai}(\lambda) p_\lambda(\boldsymbol{x}) = \sum_{\rho \vdash 2n} \frac{n(2n-\ell(\rho))!}{(n-\ell(\rho)+1)!} 2^{\ell(\rho)-n} m_\rho(\boldsymbol{x}).$$

Let us make three remarks on this statement. First, together with the trivial fact that $\mathrm{Ai}(\lambda) = 0$ if λ is a partition of an odd number, it entirely determines the numbers $\mathrm{Ai}(\lambda)$ (as power sums form a basis of the ring of symmetric functions). Second, it implies the Harer-Zagier summation formula (which can be recovered by setting $\boldsymbol{x} = (1, \dots, 1, 0, \dots)$ with exactly x times the value 1). Third, it admits an equivalent combinatorial formulation, that we shall present now.

As in the previous subsection, we shall consider colored maps, that is maps whose vertices are colored with numbers from 1 to r, each color being used at least one. For such a map, one can consider its colored vertex degree distribution: by definition, it is the composition $\boldsymbol{I} = (I_1, \dots, I_r)$ such that I_k is the sum of the degrees of the vertices of color k.

We denote by $\mathrm{AiC}(\boldsymbol{I})$ the number of colored maps with colored vertex degree distribution $\boldsymbol{I}$. Then, using the tools of [24, Section 2], one can easily show that Proposition 12 is equivalent to the following statement, that we can prove using our main bijection.

Proposition 13. *For any composition $\boldsymbol{I}$ of $2n$ of length r,*

$$\mathrm{AiC}(\boldsymbol{I}) = \frac{n(2n-r)!}{(n-r+1)!} 2^{r-n}.$$

Proof. We shall first consider the case where $\boldsymbol{I}$ has length $n+1$. This means that we count rooted unicellular maps with n edges and vertices of $n+1$ colors. But a unicellular map is necessarily connected and, hence, has at most $n+1$ vertices. Therefore, we are counting rooted unicellular maps with $n+1$ labelled vertices, that is, rooted plane trees with labelled vertices of prescribed degrees. In that case, one can show that $\mathrm{AiC}(\boldsymbol{I}) = 2n!$ by several methods (e.g., the cycle lemma, or Pitman's aggregation process [20], or the more recent method by Bernardi and Morales [6]). We give here a short proof by induction.Note that an unrooted vertex-labelled plane tree cannot have any symmetry and thus can always be rooted in $2n$ ways. So we shall rather compute the number $\mathrm{UT}(\boldsymbol{I})$ of unrooted vertex-labelled plane trees with degree distribution $\boldsymbol{I}$.

We shall prove by induction that $\mathrm{UT}(\boldsymbol{I}) = (n-1)!$.

For $n = 1$, the only possibility is $\boldsymbol{I} = (1, 1)$ and there is only one tree with such degree distribution: ①–②. Thus $\mathrm{UT}((1, 1))$.

Let $\boldsymbol{I}$ be a composition of $2n$ of length $n+1$. This composition must contain a part equal to 1 and without loss of generality ($\mathrm{UT}(\boldsymbol{I})$ is invariant by permutation of the parts of $\boldsymbol{I}$), we may assume that $I_{n+1} = 1$. This means that we are counting trees T, in which the vertex labelled $n+1$ is a leaf. Denote by j the label of the vertex to which this leaf is attached. Then, removing the leaf $n+1$ from T, we obtained a tree T' of degree distribution $\boldsymbol{I}^{(j)} = (I_1, \dots, I_j - 1, \dots, I_n)$. For each such tree T', a new leaf labelled $n+1$ can be attached to the vertex j in $I_j - 1$ ways (recall that we are dealing with plane trees). Therefore,

$$\mathrm{UT}(\boldsymbol{I}) = \sum_{j=1}^{n} (I_j - 1) \cdot \mathrm{UT}(\boldsymbol{I}^{(j)}).$$

From this induction relation, it is immediate to see that $\mathrm{UT}(\boldsymbol{I}) = (n-1)!$ for any composition $\boldsymbol{I}$ of $2n$ of length $n+1$. Considering rooted trees instead of unrooted trees, we get that, in this case

$$\mathrm{AiC}(\boldsymbol{I}) = 2n!.$$

Turn back to the general case. Let $\boldsymbol{I}$ be a composition of $2n$ of length smaller than $n+1$. Let us consider a composition $\boldsymbol{H}$ of length $n+1$ refining $\boldsymbol{I}$. and consider a labelled plane tree T whose vertex degree distribution is $\boldsymbol{H}$ (because of the labels, this distribution is ordered and can be seen as a composition).

Let us color our tree as follows: we give color s to a vertex if the corresponding part of $\boldsymbol{H}$ is contained in the s-th part of I. Since the tree is labelled, vertices with the same color are totally ordered. Hence if we add the data of a sign per color (2^r choices for all signs), using Lemma 3, we can see the vertices with the same color as endowed with a C-permutation.

Putting all these C-permutations together, we obtain a C-permutation of the vertices of the tree T, which has the following property: vertices in the same cycle always have the same color. Applying our main bijection (Theorem 5), we obtain a unicellular map. The vertices of this map have a canonical coloration, as each vertex corresponds to a cycle of the C-permutation. By construction, this colored map has colored degree distribution $\boldsymbol{I}$.

To sum up, by Theorem 5 and the construction above, each colored unicellular map with colored degree distribution $\boldsymbol{I}$ can be obtained in 2^{n+1} different ways from

- a labelled plane tree T of vertex degree distribution given by $\boldsymbol{H}$ for *some* refinement $\boldsymbol{H}$ of $\boldsymbol{I}$ of length $n+1$;
- the assignment of a sign to each color.

The number of possible signs is always 2^r, so this yields a constant factor. For a given composition $\boldsymbol{H}$, the number of corresponding trees is $2n!$ (as seen above); in particular, it does not depend on $\boldsymbol{H}$. Besides, looking at compositions as descent sets, it is easy to see that there are

$$\binom{2n-\ell(\boldsymbol{I})}{n+1-\ell(\boldsymbol{I})}$$

refinements $\boldsymbol{H}$ of $\boldsymbol{I}$ of length $n+1$. Finally, by Theorem 5, we get:

$$2^{n+1}\mathrm{AiC}(\boldsymbol{I}) = 2^r \cdot \binom{2n-\ell(\boldsymbol{I})}{n+1-\ell(\boldsymbol{I})} \cdot 2n!,$$

which simplifies to the claimed formula. □

3.4.4. *A refinement.* A. Morales and E. Vassilieva [24] have established a very elegant summation formula for bipartite maps, counted with respect to their degree distributions, which can be viewed as a refinement of Jackson's summation formula (indeed, it is an easy exercise to recover Jackson's summation formula out of it). It can be noted that this formula is to Jackson's summation formula what Proposition 12 is to the Harer-Zagier summation formula:

Proposition 14 (Morales and Vassilieva [24, Theorem 1])**.** *Let m_λ and p_ρ be the monomial and power sum bases of the ring of symmetric functions and $\boldsymbol{x}$ and $\boldsymbol{y}$ two infinite sets of variables. Then, for any $n \geq 1$,*

$$\sum_{\lambda,\mu \vdash n} \mathrm{Bi}(\lambda,\mu) p_\lambda(\boldsymbol{x}) p_\mu(\boldsymbol{y}) = \sum_{\rho,\nu \vdash n} \frac{n(n-\ell(\rho))!(n-\ell(\nu))!}{(n+1-\ell(\rho)-\ell(\nu))!} m_\rho(\boldsymbol{x}) m_\nu(\boldsymbol{y}).$$

The original proof given in [24] goes through a complicated bijection with newly introduced objects called *thorn trees* by the authors. The bijective method in [2] (which is well adapted to summation formulas) also makes it possible to get the formula. And a short non-bijective proof has been given recently in [31] using characters of the symmetric groups and Schur functions. We explain here how this result can be recovered from our bijection. The proof is very similar to the one of Goupil-Schaeffer's formula.

Let us first recall that, as Proposition 12, Proposition 14 can be reformulated in purely combinatorial terms (without symmetric functions) using colored maps.

By definition here, a bipartite unicellular map is *colored* by associating to each white (resp. black) vertex a color in $[\![1,\ell_w]\!]$ (resp. $[\![1,\ell_b]\!]$), each color between 1 and ℓ_w (resp. ℓ_b) being chosen at least once (note: we always think of the color r of a white vertex as *different* from the color r of a black vertex). To a colored bipartite map with n edges one can associate its *colored degree distribution*, that is, the pair $(\boldsymbol{I},\boldsymbol{J})$ of compositions of n, where the k-th part of $\boldsymbol{I}$ (resp. of $\boldsymbol{J}$) is the sum of the degrees of the white (resp. black) vertices with color k.

We denote by $\mathrm{BiC}(\boldsymbol{I},\boldsymbol{J})$ the number of colored bipartite unicellular maps of colored degree distribution $(\boldsymbol{I},\boldsymbol{J})$. Then Proposition 14 is equivalent to the following statement [24, paragraph 2.4]:

Proposition 15. *For any compositions $\boldsymbol{I}$ and $\boldsymbol{J}$ of the same integer n which satisfy $\ell(\boldsymbol{I})+\ell(\boldsymbol{J}) \leq n+1$, one has*

$$\mathrm{BiC}(\boldsymbol{I},\boldsymbol{J}) = \frac{n(n-\ell(\boldsymbol{I}))!(n-\ell(\boldsymbol{J}))!}{(n+1-\ell(\boldsymbol{I})-\ell(\boldsymbol{J}))!}.$$

Proof. The proof of this proposition is very similar to the one of Proposition 13. First, in the case where $\ell(\boldsymbol{I})+\ell(\boldsymbol{J}) = n+1$, a proof by induction similar to the one in Proposition 13 yields

$$\mathrm{BiC}(\boldsymbol{I},\boldsymbol{J}) = n(\ell(\boldsymbol{I})-1)!(\ell(\boldsymbol{J})-1)! = n(n-\ell(\boldsymbol{I}))!(n-\ell(\boldsymbol{J}))!.$$

We now turn to the general case. Let us consider two compositions $\boldsymbol{H}$ and $\boldsymbol{K}$ which refine respectively $\boldsymbol{I}$ and $\boldsymbol{J}$, and such that $\ell(\boldsymbol{H})+\ell(\boldsymbol{K}) = n+1$. A part of $\boldsymbol{H}$ (resp. of $\boldsymbol{K}$) is said to have color r if it is contained in the r-th part of $\boldsymbol{I}$ (resp. of $\boldsymbol{J}$).

Consider a labelled bipartite tree T whose white vertex degrees (in the order given by the labels) follow the composition $\boldsymbol{H}$ and whose black vertex degrees follow the composition $\boldsymbol{K}$. A black (resp. white) vertex is said to have color r if the corresponding part of $\boldsymbol{H}$ (resp. of $\boldsymbol{K}$) has color r. Since the tree is labelled, the white (resp. black) vertices with the same color r are totally ordered. Hence if we add the data of a sign per color ($2^{\ell(I)+\ell(J)}$ choices for all signs), using Lemma 3, we can see the vertices with the same color as endowed with a C-permutation.

Putting all these C-permutations together, we obtain a C-permutation of the vertices of the tree T, which has the following property: the vertices in the same cycle always have the same color. Applying our main bijection (Theorem 5), we obtain a bipartite unicellular map. The vertices of this map have a canonical coloration, as each vertex corresponds to a cycle of the C-permutation. By construction, this colored map has colored degree distribution $(\boldsymbol{I},\boldsymbol{J})$.

To sum up, by Theorem 5 and the construction above, each colored bipartite unicellular map with colored degree distribution $(\boldsymbol{I},\boldsymbol{J})$ can be obtained in 2^{n+1} different ways from

- a labelled bipartite tree T of white (resp. black) vertex degree given by $\boldsymbol{H}$ (resp. $\boldsymbol{K}$) for *some* refinements $\boldsymbol{H}$ and $\boldsymbol{K}$ with $\ell(\boldsymbol{H})+\ell(\boldsymbol{K})=n+1$;
- the assignment of a sign to each color.

The number of possible signs is always $2^{\ell(I)+\ell(J)}$, so this yields a constant factor. For given compositions $\boldsymbol{H}$ and $\boldsymbol{K}$, the number of corresponding trees is

$$n(\ell(\boldsymbol{H})-1)!(\ell(\boldsymbol{K})-1)!$$

Thus we have to count the number of refinements $\boldsymbol{H}$ (resp. $\boldsymbol{K}$) of $\boldsymbol{I}$ (resp. $\boldsymbol{J}$) with a given value ℓ of $\ell(\boldsymbol{H})$ (resp. m of $\ell(\boldsymbol{K})$). It is easily seen to be equal to

$$\binom{n-\ell(\boldsymbol{I})}{\ell-\ell(\boldsymbol{I})} \text{ (resp. } \binom{n-\ell(\boldsymbol{J})}{m-\ell(\boldsymbol{J})} \text{).}$$

Finally, by Theorem 5, we get:

$$2^{n+1}\mathrm{BiC}(\boldsymbol{I},\boldsymbol{J}) = 2^{\ell(\boldsymbol{I})+\ell(\boldsymbol{J})} \sum_{\substack{\ell+m=n+1\\ \ell\geq\ell(\boldsymbol{I}),\ m\geq\ell(\boldsymbol{J})}} n(\ell-1)!(m-1)!\binom{n-\ell(\boldsymbol{I})}{\ell-\ell(\boldsymbol{I})}\binom{n-\ell(\boldsymbol{J})}{m-\ell(\boldsymbol{J})}.$$

Denoting $h=n+1-\ell(\boldsymbol{I})-\ell(\boldsymbol{J})$ and setting $h_1=\ell-\ell(\boldsymbol{I}), h_2=m-\ell(\boldsymbol{J})$ in the summation index, the right-hand side of the previous equation writes as:

$$2^{n+1}\mathrm{BiC}(\boldsymbol{I},\boldsymbol{J}) = 2^{\ell(I)+\ell(J)} \sum_{h_1+h_2=h} n(\ell(\boldsymbol{I})+h_1-1)!(\ell(\boldsymbol{J})+h_2-1)!\binom{n-\ell(\boldsymbol{I})}{h_1}\binom{n-\ell(\boldsymbol{J})}{h_2}.$$

But the relation $\ell(\boldsymbol{I})+\ell(\boldsymbol{J})+h_1+h_2=n+1$ implies that $(\ell(\boldsymbol{J})+h_2-1)!\binom{n-\ell(\boldsymbol{I})}{h_1}=\frac{(n-\ell(\boldsymbol{I}))!}{h_1!}$ and $(\ell(\boldsymbol{I})+h_1-1)!\binom{n-\ell(\boldsymbol{J})}{h_2}=\frac{(n-\ell(\boldsymbol{J}))!}{h_2!}$. Plugging this in the expression above, we get

$$\begin{aligned} 2^{n+1}\mathrm{BiC}(\boldsymbol{I},\boldsymbol{J}) &= 2^{\ell(I)+\ell(J)}\cdot n\cdot(n-\ell(\boldsymbol{I}))!\cdot(n-\ell(\boldsymbol{J}))! \sum_{h_1+h_2=h}\frac{1}{h_1!h_2!} \\ &= 2^{\ell(I)+\ell(J)}\cdot n\cdot(n-\ell(\boldsymbol{I}))!\cdot(n-\ell(\boldsymbol{J}))!\frac{2^h}{h!}. \end{aligned}$$

The powers of 2 cancel each other and we get the desired result. □

3.5. **Covered maps, shuffles, and an identity of** [3]. *Covered maps* were introduced in [3] as an extension of the notion of tree-rooted map (map equipped with a spanning tree). A covered map of genus g is a rooted map M of genus g, not necessarily unicellular, equipped with a distinguished connected subgraph S (with the same vertex set as M) having the following property:

> *viewed as a map, S is a unicellular map, possibly of a different genus than M.*

Here, in order to view S "as a map", we equip it with the map structure induced by M: the clockwise ordering of half-edges of S around each vertex is defined as the restriction of the clockwise ordering in M (see [3] for details). The genus g_1 of S is an element of $[\![0,g]\!]$. For example, $g_1=0$ if and only if S is a spanning tree of M. In general, we say that the covered map (M,S) has *type* (g,g_1).

Covered maps have an interesting duality property that generalizes the existence of dual spanning trees in the planar case: namely, each covered map (M,S) of type

(g, g_1) has a dual covered map (M^*, S') of type (g, g_2) with $g_1 + g_2 = g$. By extending ideas of Mullin [25], it is not difficult to describe the covered map M as a "shuffle" of the two unicellular maps S and S', see [3]. It follows that the number $\mathrm{Cov}_{g_1,g_2}(n)$ of covered maps of type $(g_1 + g_2, g_1)$ with n edges can be expressed as the following shuffle-sum [3, eq. (6)]:

$$\mathrm{Cov}_{g_1,g_2}(n) = \sum_{n_1+n_2=n} \binom{2n}{2n_1} \epsilon_{g_1}(n_1)\epsilon_{g_2}(n_2). \tag{8}$$

In the case $g_1 = g_2 = 0$, this sum simplifies thanks to the Chu-Vandermonde identity, and we have the remarkable result due to Mullin [25] (see [1] for a bijective proof):

$$\mathrm{Cov}_{0,0}(n) = \mathrm{Cat}(n)\mathrm{Cat}(n + 1). \tag{9}$$

The main enumerative result of the paper [3] is a generalisation of (9) to any genus, obtained via a difficult bijection:

Proposition 16 (Bernardi and Chapuy, [3])**.** *For all $n \geq 1$ and $g \geq 0$, the number $\mathrm{Cov}_g(n) = \sum_{g_1+g_2=g} \mathrm{Cov}_{g_1,g_2}(n)$ of covered maps of genus g with n edges is equal to:*

$$\mathrm{Cov}_g(n) = \mathrm{Cat}(n)\mathrm{Bip}_g(n + 1),$$

where $\mathrm{Bip}_g(n+1)$ is the number of bipartite unicellular maps of genus g with $n+1$ edges. Equivalently, the following identity holds:

$$\sum_{g_1+g_2=g} \sum_{n_1+n_2=n} \binom{2n}{2n_1} \epsilon_{g_1}(n_1)\epsilon_{g_2}(n_2) = \mathrm{Cat}(n)\mathrm{Bip}_g(n + 1). \tag{10}$$

Proof. We denote as before by $c_g(m)$ the number of C-permutations of genus g of a set of m elements. By our main result, Theorem 5, the left-hand side of (10) can be rewritten as:

$$2^{-n-2} \sum_{g_1+g_2=g} \sum_{n_1+n_2=n} \binom{2n}{2n_1} c_{g_1}(n_1 + 1)c_{g_2}(n_2 + 1)\mathrm{Cat}(n_1)\mathrm{Cat}(n_2).$$

We now observe that:

$$\binom{2n}{2n_1}\mathrm{Cat}(n_1)\mathrm{Cat}(n_2) = \mathrm{Cat}(n)\mathrm{Nar}(n_1 + 1, n_2 + 1; n + 1)$$

where as before the Narayana number $\mathrm{Nar}(i, j; n)$ is the number of bipartite plane trees with n edges, i black vertices and j white vertices (this last equality follows directly from the explicit expressions of Catalan and Narayana numbers; an interpretation in terms of planar tree-rooted maps is given by the bijection of [1]). Therefore we have:

$$\mathrm{Cov}_g(n) = 2^{-n-2}\mathrm{Cat}(n) \sum_{g_1+g_2=g} \sum_{n_1+n_2=n} c_{g_1}(n_1+1)c_{g_2}(n_2+1)\mathrm{Nar}(n_1+1, n_2+1; n+1).$$

Now, the double-sum in this equation is equal to the number of bipartite C-decorated trees (that is, bipartite trees equipped with a C-permutation of the vertices that stabilizes each color class) with $n + 1$ edges and genus g: indeed in the double-sum, the quantities g_1 and $n_1 + 1$ can be interpreted respectively as the genus of the restriction of the C-permutation to black vertices of the tree, and as the number of black vertices in the tree. By our main result, Theorem 5, this double-sum is therefore equal to $2^{n+2}\mathrm{Bip}_g(n + 1)$, which proves (10). $\square$

The proof above and (8) also show the following fact. Let G_1 be the genus of the submap S in a covered map (M,S) of genus g with n edges chosen uniformly at random, and let $G_\circ$ be the genus of the restriction to white vertices of the C-permutation in a bipartite C-decorated tree of genus g with $n+1$ edges chosen uniformly at random. Then the random variables G_1 and $G_\circ$ have the same distribution.

It is possible to prove that, when g is fixed and n tends to infinity, the variable $G_\circ$ is close to a binomial random variable $B(g,1/2)$: the idea behind this property is that a random bipartite tree with $n+1$ edges has about $n/2+O(\sqrt{n})$ vertices of each color with high probability, and that with high probability the C-permutation of its vertices is made of g cycles of length 3, that independently "fall" into each of the two color classes with probability 1/2. Giving a proper proof of these elementary statements would lead us too far from our main subject, so we leave to the reader the details of a proof along these lines of the following fact, which was proved in [3] with no combinatorial interpretation:

Proposition 17 ([3])**.** *Let $g \geq g_1 \geq 0$. When n tends to infinity, the probability that a covered map of genus g with n edges chosen uniformly at random has type (g,g_1) tends to $2^{-g}\binom{g}{g_1}$.*

To conclude this section, we mention that, in [3], refined results were given that take more parameters into account (*e.g.*, the number of vertices and faces of the covered map). These extensions can be proved exactly in the same way as Proposition 16, but we do not state them explicitly here, for the sake of brevity.

4. Computing Stanley character polynomials

4.1. **Formulation of the problem.** We now consider the following enumerative problem. For n a fixed integer, we would like to compute the generating series

$$F_n(p_1,p_2,\cdots;q_1,q_2,\cdots) = \sum_{(M,\varphi)} \mathrm{wt}(M,\varphi)$$

of pairs (M,φ) where M is a rooted bipartite unicellular map with n edges, and φ is a mapping from the vertex set V_M of M to positive integers, satisfying the following *order condition*:

> *for each edge e of M, one has $\varphi(b_e) \geq \varphi(w_e)$, where b_e and w_e are respectively the black and white extremities of e.*

The weight of such a pair is $\mathrm{wt}(M,\varphi) := \prod_{v\in V_M^\circ} p_{\varphi(v)} \prod_{v\in V_M^\bullet} q_{\varphi(v)}$, where $V_M^\bullet$ and V_M° are respectively the sets of black (resp. white) vertices of M.

Our motivation comes from representation theory of the symmetric group. This topic is linked to map enumeration by the following formula conjectured in [30] and proved in [14]. Let $\boldsymbol{p} = p_1,\cdots,p_r$ and $\boldsymbol{q} = q_1,\cdots,q_r$ be two finite lists of positive integers of the same length. Then the evaluation of the generating series considered above is equal to

$$F_n(p_1,\cdots,p_r,0,\cdots;q_1,\cdots,q_r,0,\cdots) = L(L-1)\cdots(L-n+1)\hat{\chi}^\lambda((1\ 2\ \cdots\ n)), \tag{11}$$

where:

- λ is the partition with p_1 parts equal to $q_1+\cdots+q_r$, p_2 parts equal to $q_2+\cdots+q_r$, and so on...

- $L = \sum_{1 \le i \le j \le r} p_i q_j$ is the number of boxes of λ;
- $\hat{\chi}^\lambda$ is the normalized character of the irreducible representation of S_L associated to λ;
- $(1\ 2\ \cdots\ n)$ is an n-th cycle seen as a permutation of S_L (if $n > L$, it is not defined but, as the numerical factor is 0, it is not a problem).

Remark 2. In [30, 14], this formula is stated under a slightly different form. We call G_n the same generating series as F_n except that the order condition is replaced by the following *maximum condition*:

> *for each black vertex b, one has $\varphi(b) = \max \varphi(w)$, where the maximum is taken over all white neighbours w of b.*

Then the main theorem of [14] states that

$$G_n(p'_1, \cdots, p'_r, 0, \cdots; q'_1, \cdots, q'_r, 0, \cdots) = L(L-1)\cdots(L-n+1)\hat{\chi}^\lambda((1\ 2\ \cdots\ n)),$$

where everything is defined as above except that

> *λ is the partition with p'_1 parts equal to q'_1, p'_2 parts equal to q'_2, and so on...*

This result is clearly equivalent to (11) by setting:

$$\forall i \ge 1, \begin{cases} p_i = p'_i \\ q_i = q'_i - q'_{i+1} \end{cases}.$$

4.2. **A new expression for F_n.** Our main bijection allows us to express the generating series F_n in terms of the corresponding generating series for plane trees:

$$R_{n+1}(\boldsymbol{p}, \boldsymbol{q}) = \sum_{(T,\varphi)} \mathrm{wt}(T, \varphi),$$

where the sum runs over all pairs (T, φ), T being a plane tree with n edges and φ a function $V_T \to \mathbb{N}$ satisfying the order condition.

The strange notation R_{n+1} comes from the following fact: A. Rattan has proved [27] that this generating series is the $n+1$-th free cumulant R_{n+1} of the transition measure of the Young diagram λ (λ stands here for the Young diagram defined in terms on $\boldsymbol{p}$ and $\boldsymbol{q}$ in the previous paragraph). Free cumulants have become in the last few years an important tool in (asymptotic) representation theory of the symmetric groups, see for example the work of P. Biane [8].

Let us define an operator D by[4]:

$$D(x^k) := \sum_{g \ge 0} c_g(k) x^{k-2g} = k! \sum_{r=1}^{k} 2^r \binom{k-1}{r-1} \binom{x}{r},$$

D being extended multiplicatively to monomials in distinct variables, and then extended linearly to multivariate polynomials and series (in particular, series in the variables $\boldsymbol{p}$ and $\boldsymbol{q}$).

Theorem 18. *For any $n \ge 1$, one has $2^{n+1} F_n = D(R_{n+1})$.*

Proof. A pair (M, φ) as above corresponds by the bijection of Theorem 5 to a bipartite C-decorated tree T, together with a function $\varphi : V_T \to \mathbb{N}$ which fulfills the order condition and such that all vertices in a given cycle have the same image by φ. Equivalently, we choose the tree T, a function $\varphi : V_T \to \mathbb{N}$ and then, for each

[4] The second equality is obtained by similar arguments as in the proof of Proposition 10.

$i \geq 1$ a C-permutation of the white (resp. black) vertices of value i. The result follows directly. □

The free cumulant R_{n+1} is the compositional inverse of an explicit series [27]. Hence Theorem 18 gives an efficient, easily implemented way of computing Stanley character polynomials F_n.

5. Counting 3-constellations

5.1. **Constellations and the Poulalhon-Schaeffer formula.** Constellations are a family of colored maps, depending on an integer parameter $m \geq 2$, that are in bijection with factorizations of a permutation into m factors. We refer to [22, Chapter 1] for a general discussion on constellations, and in particular to Section 1.6.2 of this book for the correspondence between the factorization viewpoint and the map-theoretic perspective that we adopt here (see also [7, Section 2], or the introduction of [9]). For $m = 2$, constellations are in bijection with bipartite maps, which are well-known to be in bijection with factorizations of permutations into 2 factors [13].

Definition 2. *An m-constellation is a map with circle and square vertices such that:*

- *(i) the circle vertices are colored with m colors $1, 2, \dots, m$;*
- *(ii) all edges have one circle and one square extremity;*
- *(iii) each square vertex is linked to exactly one circle vertex of each color;*
- *(iv) moreover, the circle vertices around each square vertex appear counterclockwise in the cyclic order $1, 2, \dots, m$.*

A constellation is rooted *if we distinguish a corner of a circle vertex of color* 1. *Unless mentioned explicitly, all constellations considered will be rooted.*

The size of an m-constellation is its number of square vertices.

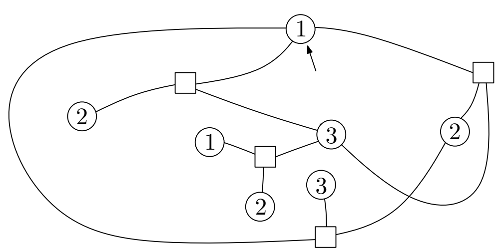


FIGURE 5. A rooted 3-constellation of size 4 (the root corner is pointed with an arrow). This 3-constellation has genus 1 and is unicellular.

Note that several papers, *e.g.* [9, 10, 11, 7], use a different but equivalent definition of constellations in terms of maps, where square vertices are replaced by "black faces" of degree m. For the purpose of using the bijection of Section 2.2, the definition we use here will be much more convenient.

Fix an m-constellation of size n. The sequence of the degrees of its circle vertices of color i, taken in decreasing order, forms a partition $\lambda^{(i)}$ of size n. The list $\lambda^{(1)}, \dots, \lambda^{(m)}$ is called the *multitype* of the constellation.

For unicellular m-constellations, the Euler formula links the genus and the multitype (here, ℓ_i is the length of $\lambda^{(i)}$):

$$2g = n(m-1) + 1 - \sum_{i=1}^{m} \ell_i.$$

Using algebraic tools, D. Poulalhon and G. Schaeffer have given a general formula for the number of unicellular m-constellations of size n [26, Theorem 1] with a given multitype. Though explicit, their formula requires quite heavy notations to be stated, therefore we present here only the case $m = 3$, which is the only case we are able to attack with our combinatorial tools.

For a partition λ of length ℓ and a non-negative integer g, we denote

$$\begin{aligned} a(\lambda) &= \prod m_i(\lambda)!; \\ S_g(\lambda) &= (\ell + 2g - 1)! \sum_{p_1+\cdots+p_\ell = g} \prod_{i=1}^{\ell} \frac{1}{2p_i+1} \binom{\lambda_i - 1}{2p_i}. \end{aligned}$$

Theorem 19 (Poulalhon and Schaeffer, 2002)**.** *Let $\lambda^{(1)}$, $\lambda^{(2)}$ and $\lambda^{(3)}$ be three partitions of lengths ℓ_1, ℓ_2 and ℓ_3 and of the same size n, such that*

$$g = 1/2 \cdot (2n + 1 - \ell_1 - \ell_2 - \ell_3)$$

is a non-negative integer. Then the number $c_{\boldsymbol{\lambda}}$ of 3-constellations of multitype $(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$ is given by the formula

$$\begin{aligned} c_{\boldsymbol{\lambda}} = \frac{n^2}{2^{2g} a(\lambda^{(1)}) a(\lambda^{(2)}) a(\lambda^{(3)})} \sum_{g_0+g_1+g_2+g_3 = g} \Bigg[(g_0!)^2 \binom{n - \ell_1 - 2g_1}{g_0} \\ \cdot \binom{n - \ell_2 - 2g_2}{g_0} \binom{n - \ell_3 - 2g_3}{g_0} S_{g_1}(\lambda^{(1)}) S_{g_2}(\lambda^{(2)}) S_{g_3}(\lambda^{(3)}) \Bigg]. \end{aligned}$$

The planar case ($g = 0$) of this theorem:

$$c_{\boldsymbol{\lambda}} = \frac{n^2}{a(\lambda^{(1)}) a(\lambda^{(2)}) a(\lambda^{(3)})} (\ell_1 - 1)!(\ell_2 - 1)!(\ell_3 - 1)!, \tag{12}$$

has been proved earlier by I. Goulden and D. Jackson [16, Theorem 3.2] and can be handled in a purely combinatorial way [7].

Finding a combinatorial proof of Theorem 19 for higher genus is an open problem (and seems difficult, due to the complexity of the formula). We did not succeed in solving this problem but we shall present two results in this direction.

5.2. **Refined enumeration of quasi-constellations.** Our bijection preserves the underlying multi-graph of a unicellular map, but not the cyclic order around vertices. Therefore the last condition in the definition of constellations is hard to handle with our method.

This paragraph is devoted to the proof of a refined enumeration formula for new objects that we call 3-*quasi-constellations*, which are defined by the same conditions as constellations, except that condition (iv) is dropped.

The formula obtained is surprisingly close to the one for constellations, although we are not able to explain this phenomenon.

Proposition 20. *Let $\lambda^{(1)}$, $\lambda^{(2)}$ and $\lambda^{(3)}$ be three partitions of lengths ℓ_1, ℓ_2 and ℓ_3 and of the same size n, such that $g = 1/2 \cdot (2n+1-\ell_1-\ell_2-\ell_3)$ is a non-negative integer. Then the number $\tilde{c}_{\boldsymbol{\lambda}}$ of 3-quasi-constellations of multitype $(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$ is given by the formula*

$$\tilde{c}_{\boldsymbol{\lambda}} = \frac{n \cdot 2^n}{2^{2g} a(\lambda^{(1)}) a(\lambda^{(2)}) a(\lambda^{(3)})} \sum_{g_0+g_1+g_2+g_3=g} (n-g_0) \Bigg[(g_0!)^2 \binom{n-\ell_1-2g_1}{g_0} \cdot \binom{n-\ell_2-2g_2}{g_0} \binom{n-\ell_3-2g_3}{g_0} S_{g_1}(\lambda^{(1)}) S_{g_2}(\lambda^{(2)}) S_{g_3}(\lambda^{(3)}) \Bigg].$$

As in the proof of Proposition 9, we prefer to work with *labelled* 3-quasi-constellations (circle vertices of each color are labelled separately, square vertices are *not* labelled). The multitype of such an object is a triple of compositions with obvious definition. We denote by $\hat{c}(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{L})$ the number of labelled 3-quasi-constellations of multitype $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{L})$, where $\boldsymbol{I}$, $\boldsymbol{J}$ and $\boldsymbol{L}$ are three compositions of the same size n. From now on, ℓ_1, ℓ_2 and ℓ_3 will denote the respective lengths of the compositions $\boldsymbol{I}$, $\boldsymbol{J}$ and $\boldsymbol{L}$. Besides, we use the standard index notation I_i, J_i and L_i for the i-th component of these compositions.

In genus 0 and size n, there are clearly 2^n times more 3-quasi-constellations than 3-constellations (indeed in the planar case, cyclically reordering the neighbours around a vertex keeps the property of being unicellular; this does not hold in higher genus). Hence, using Equation (12), we obtain in that case

$$\hat{c}(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{L}) = 2^n n^2 (\ell_1-1)!(\ell_2-1)!(\ell_3-1)!. \tag{13}$$

If we apply our main bijection to a 3-(quasi-)constellation, the tree object is not necessarily a planar 3-quasi-constellation. Indeed, one can get square vertices of degree 1. Therefore, we need to introduce the concept of prickly planar 3-quasi-constellations.

Definition 3. *A* prickly planar *3-quasi-constellation is a rooted tree with circle and square vertices such that:*

(i) the circle vertices are colored with 3 colors $1,2,3$;
(ii) all edges have one circle and one square extremity;
(iii) each square vertex is either a leaf or linked to exactly one circle vertex of each color;
(iv) the number of square leaves linked to vertices of color 1,2, 3 are the same (this number g_0 will be called the prickling number*).*

The size n of such objects is one third of the number of edges. For a labelled object, its multitype $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{L})$ is defined as for (quasi-)constellations.

Planarity is equivalent to the relation:

$$\ell_1 + \ell_2 + \ell_3 = 2n - 2g_0 + 1$$

Lemma 21. *The number of labelled prickly planar 3-quasi-constellations of multitype $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{L})$ and prickling number g_0 is*

$$n(n-g_0)2^{n-g_0} \binom{n-\ell_1}{g_0} \binom{n-\ell_2}{g_0} \binom{n-\ell_3}{g_0} (\ell_1-1)!(\ell_2-1)!(\ell_3-1)!$$

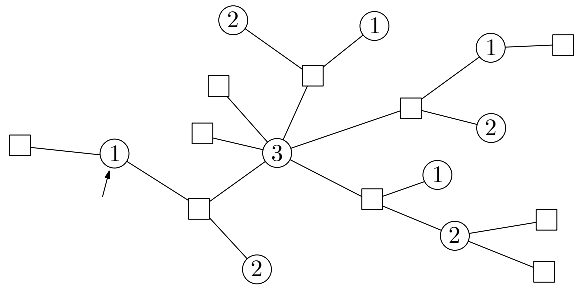


FIGURE 6. A prickly planar 3-quasi-constellation of size $n = 6$, with $l_1 = 4$, $l_2 = 4$, $l_3 = 1$. The prickling number is $g_0 = 2$.

Proof. It is easier in the proof of this lemma to work with *unrooted* labelled quasi-constellations. Note that a planar (vertex-)labelled tree has no symmetry, hence, dealing with unrooted or rooted objects only changes the counting coefficients by constant explicit factors (this would not be true for higher genus objects!).

Consider an (unrooted) prickly planar 3-quasi-constellation of multitype $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{L})$ and prickling number g_0. We denote by a_i (resp. b_i and c_i) the number of square leaves attached to the circle vertex labelled i of color 1 (resp 2 and 3). If we erase these leaves we get an unrooted planar 3-quasi-constellation of size $n - g_0$ and multitype $(\boldsymbol{I} - \boldsymbol{a}, \boldsymbol{J} - \boldsymbol{b}, \boldsymbol{L} - \boldsymbol{c})$, where, by definition, $\boldsymbol{I} - \boldsymbol{a} = (I_1 - a_1, \dots, I_{\ell_1} - a_{\ell_1})$ and similar definitions hold for $\boldsymbol{J} - \boldsymbol{b}$ and $\boldsymbol{L} - \boldsymbol{c}$.

The number of unrooted planar 3-quasi-constellation of multitype $(\boldsymbol{I} - \boldsymbol{a}, \boldsymbol{J} - \boldsymbol{b}, \boldsymbol{L} - \boldsymbol{c})$ is, by equation (13) (beware of the rooting, which yields a factor $n - g_0$):

$$2^{n-g_0}(n - g_0)(\ell_1 - 1)!(\ell_2 - 1)!(\ell_3 - 1)!.$$

If we want to recover the prickly planar 3-quasi-constellation, one has to remember for each circle vertex of color 1 (resp. 2, 3) where to add the a_i (resp. b_i, c_i) square leaves. This gives

$$\prod_{i=1}^{\ell_1} \binom{I_i - 1}{a_i} \prod_{i=1}^{\ell_2} \binom{J_i - 1}{b_i} \prod_{i=1}^{\ell_3} \binom{K_i - 1}{c_i}$$

choices (working with unrooted objects is crucial here). Finally the number of unrooted prickly planar 3-quasi-constellation of multitype $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{L})$ and prickling number g_0 is

$$2^{n-g_0}(n - g_0)(\ell_1 - 1)!(\ell_2 - 1)!(\ell_3 - 1)!$$
$$\cdot \sum_{\substack{a_1,\dots,a_{\ell_1} \\ a_1+\dots+a_{\ell_1}=g_0}} \prod_{i=1}^{\ell_1} \binom{I_i - 1}{a_i} \cdot \sum_{\substack{b_1,\dots,b_{\ell_2} \\ b_1+\dots+b_{\ell_2}=g_0}} \prod_{i=1}^{\ell_2} \binom{J_i - 1}{b_i} \cdot \sum_{\substack{c_1,\dots,c_{\ell_3} \\ c_1+\dots+c_{\ell_3}=g_0}} \prod_{i=1}^{\ell_3} \binom{L_i - 1}{c_i}.$$

The first (resp. second, third) sum corresponds to the number of ways of choosing g_0 elements among $n - \ell_1$ (resp. $n - \ell_2$, $n - \ell_3$). This yields the formula of the lemma, because every unrooted object can be rooted in n different ways. $\square$

The end of the proof of Proposition 20 is now very similar to the one of the Goupil-Schaeffer formula (Proposition 9). Therefore, we do not give all the details.

Sketch of proof of Proposition 20. Our main bijection sends a 3-quasi-constellation of multitype $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{L})$ to a C-decorated prickly planar 3-quasi-constellation of multitype $(\boldsymbol{H}, \boldsymbol{K}, \boldsymbol{M})$, where $\boldsymbol{H}$, $\boldsymbol{K}$ and $\boldsymbol{M}$ are refinements of $\boldsymbol{I}$, $\boldsymbol{J}$ and $\boldsymbol{L}$.

For given refinements $\boldsymbol{H}$, $\boldsymbol{K}$ and $\boldsymbol{M}$ of respective lengths $\ell_1 + 2g_1$, $\ell_2 + 2g_2$ and $\ell_3 + 2g_3$, the number of labelled prickly planar 3-quasi-constellations is (by Lemma 21)

$$n(n-g_0)2^{n-g_0}\binom{n-\ell_1-2g_1}{g_0}\binom{n-\ell_2-2g_2}{g_0}\binom{n-\ell_3-2g_3}{g_0} \cdot (\ell_1+2g_1-1)!(\ell_2+2g_2-1)!(\ell_3+2g_3-1)!,$$

where $g_0 = 1/2 \cdot (2n+1-\ell_1-2g_1-\ell_2-2g_2-\ell_3-2g_3)$ should be a non-negative integer and is the prickling number of the object.

We need the number of refinements $\boldsymbol{H}$ of $\boldsymbol{I}$ of length $\ell_1 + 2g_1$, where each part I_r of $\boldsymbol{I}$ corresponds to an odd number $2p_r + 1$ of parts of $\boldsymbol{H}$. It is given by

$$\sum_{p_1+\cdots+p_{\ell_1}=g_1} \prod_{r=1}^{\ell_1} \binom{I_r - 1}{2p_r}$$

and similar formulas hold for the number of refinements $\boldsymbol{K}$ and $\boldsymbol{M}$ of $\boldsymbol{J}$ and $\boldsymbol{L}$.

When we apply our bijection to a quasi-constellation, we get a C-decorated tree. As in the proof of Proposition 9, to transform this tree into a labelled prickly 3-quasi-constellations, we need:

- to choose for each cycle of circle vertices one distinguished vertex (factor $\prod_r 2p_r + 1$ for vertices of color 1 and similar factors for other colors);
- to ungroup the g_0 cycles of three square leaves (factor $1/((g_0!)^2 \cdot 2^{g_0})$);
- to forget the signs (factor $1/2^{\ell_1+\ell_2+\ell_3+n}$).

Finally, we get, by Theorem 5, that $2^{3n+1}\hat{c}(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{L})$ is

$$\sum_{g_0+g_1+g_2+g_3=g} 2^{\ell_1+\ell_2+\ell_3+n}(g_0!)^2 \cdot 2^{g_0}$$
$$\cdot \sum_{\substack{p_1+\cdots+p_{\ell_1}=g_1\\ q_1+\cdots+q_{\ell_2}=g_2\\ s_1+\cdots+s_{\ell_3}=g_3}} \prod_{r=1}^{\ell_1}\frac{1}{2p_r+1}\binom{I_r-1}{2p_r} \cdot \prod_{r=1}^{\ell_2}\frac{1}{2q_r+1}\binom{J_r-1}{2q_r} \cdot \prod_{r=1}^{\ell_3}\frac{1}{2s_r+1}\binom{L_r-1}{2s_r}$$
$$\cdot \Bigg[n(n-g_0)2^{n-g_0}\binom{n-\ell_1-2g_1}{g_0}\binom{n-\ell_2-2g_2}{g_0} \binom{n-\ell_3-2g_3}{g_0}(\ell_1+2g_1-1)!(\ell_2+2g_2-1)!(\ell_3+2g_3-1)!\Bigg],$$

which equals (using $\ell_1 + \ell_2 + \ell_3 = 2n - 2g + 1$)

$$\sum_{g_0+g_1+g_2+g_3=g} n(n-g_0)2^{4n-2g+1}(g_0!)^2$$
$$\cdot \binom{n-\ell_1-2g_1}{g_0} S_{g_1}(\lambda^{(1)}) \cdot \binom{n-\ell_2-2g_2}{g_0} S_{g_2}(\lambda^{(2)}) \cdot \binom{n-\ell_3-2g_3}{g_0} S_{g_3}(\lambda^{(3)}).$$

Using the fact that

$$\hat{c}(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{L}) = a(\lambda^{(1)})a(\lambda^{(2)})a(\lambda^{(3)}) \cdot \tilde{c}_{\lambda^{(1)},\lambda^{(2)},\lambda^{(3)}}$$

if $\lambda^{(1)}$, $\lambda^{(2)}$ and $\lambda^{(3)}$ are the sorted version of $\boldsymbol{I}$, $\boldsymbol{J}$ and $\boldsymbol{K}$, this ends the proof of Proposition 20. □

5.3. **Enumeration of constellations taking only the numbers of vertices into account.** In the previous section, we have seen that Theorem 5 is not suitable to count unicellular constellations, as it does not give any information on the rotation system of the map.

Nevertheless, it is possible to get some enumerative results on constellations not by using Theorem 5 directly, but by mimicking its proof: find a combinatorial induction for constellations and then find simpler objects with the same induction.

In this paragraph, we give a combinatorial proof of the enumeration of 3-constellations with respect to their number of circle vertices of each color.

5.3.1. *Combinatorial induction for 3-constellations.* Let us first introduce some notation: define $\mathcal{C}_{\ell_1,\ell_2,\ell_3}(n)$ as the set of 3-constellations with ℓ_1 (resp. ℓ_2 and ℓ_3) circle vertices of color 1 (resp. 2 and 3). Its cardinality is denoted $c_{\ell_1,\ell_2,\ell_3}(n)$, and the corresponding genus g is given by $\ell_1+\ell_2+\ell_3 = 2n-2g+1$. For $g=0$ it is well known that $c_{\ell_1,\ell_2,\ell_3}(n)$ is the multitype (3 types here) Narayana number:

$$c_{\ell_1,\ell_2,\ell_3}(n) = \frac{1}{n}\binom{n}{\ell_1}\binom{n}{\ell_2}\binom{n}{\ell_3}, \tag{14}$$

see for instance [6] for a combinatorial proof.

Let us apply Chapuy's bijection (recalled in Subsection 2.2) to a 3-constellation of genus g in $\mathcal{C}_{\ell_1,\ell_2,\ell_3}(n)$. Two things may happen:

- either the sliced vertex is a circle vertex, in which case we get a 3-constellation of genus $g-h$ with $2h+1$ marked circle vertices of the corresponding color (for some $h \geq 1$);
- or the sliced vertex is a square vertex. In this case, as square vertices have degree 3, the resulting map has always genus $g-1$ and has three square leaves. Erasing these leaves, we get a 3-constellation of size $n-1$ and genus $g-1$.

In the first case, Chapuy's inverse mapping works well and always produces a constellation in $\mathcal{C}_{\ell_1,\ell_2,\ell_3}(n)$.

In the second case, one has to choose where to add square leaves, that is to choose a corner c_i of a circle vertex of color i for $i=1,2,3$ ($n(n-1)^2$ choices; beware of the root). Then, when we apply Chapuy's inverse mapping, it may happen that the newly created square vertex does not fulfill condition (iv) of the definition of constellations (Definition 2).

Lemma 22. *Chapuy's inverse mapping leads to a constellation if and only if the three chosen corners appear in the cyclic order (c_1,c_2,c_3) when we turn clockwise around the unique face of the map.*

Proof. This is direct by construction: let f,f',f'' be three leaves in a unicellular map and e,e',e'' the edges incident to them, respectively. Then (e,e',e'') appear in counterclockwise order around the new vertex created by the gluing of the three leaves if and only if f,f',f'' appear in clockwise order around the face in the original map. See Subsection 2.2 (or [12, Paragraph 4.2] for a direct description of the inverse mapping). □

Lemma 23. *For each constellation of size $n-1$ and genus $g-1$, exactly $n^2(n-1)/2$ triples of corners (c_1, c_2, c_3) over $n(n-1)^2$ satisfy the condition above.*

Proof. Observe that a rooted unicellular 3-constellation of size $n-1$ has n (resp. $n-1$) corners incident to circle vertices of color 1 (resp. 2 and 3), and that clockwise around the face the colors of these $3n-2$) corners appear in the order $(1, 2, 3, 1, 2, 3, \ldots, 1, 2, 3, 1)$. From there it is a simple exercise to check the statement of the lemma. ☐

Finally, with g defined by $\ell_1+\ell_2+\ell_3+2g = 2n+1$, we get the following inductive relation (for $g > 0$):

$$2g\, c_{\ell_1,\ell_2,\ell_3}(n) = \frac{n^2(n-1)}{2} c_{\ell_1,\ell_2,\ell_3}(n-1) + \sum_{h\geq 1} \binom{\ell_1+2h}{2h+1} c_{\ell_1+2h,\ell_2,\ell_3}(n) + \binom{\ell_2+2h}{2h+1} c_{\ell_1,\ell_2+2h,\ell_3}(n) + \binom{\ell_3+2h}{2h+1} c_{\ell_1,\ell_2,\ell_3+2h}(n). \tag{15}$$

Note that the coefficients $c_{\ell_1,\ell_2,\ell_3}(n)$ are completely specified by the induction and by (14) (expression for $g = 0$).

We can define some refinement of the number $c_{\ell_1,\ell_2,\ell_3}(n)$ by the induction (with g defined as usual by $\ell_1 + \ell_2 + \ell_3 + 2g = 2n + 1$)

$$2g\, c_{\ell_1,\ell_2,\ell_3}(n; g_0) = \frac{n^2(n-1)}{2} c_{\ell_1,\ell_2,\ell_3}(n-1; g_0-1) + \sum_{h\geq 1} \binom{\ell_1+2h}{2h+1} c_{\ell_1+2h,\ell_2,\ell_3}(n; g_0) + \binom{\ell_2+2h}{2h+1} c_{\ell_1,\ell_2+2h,\ell_3}(n; g_0) + \binom{\ell_3+2h}{2h+1} c_{\ell_1,\ell_2,\ell_3+2h}(n; g_0)$$

and initial conditions that for $g = 0$, $c_{\ell_1,\ell_2,\ell_3}(n; 0) = c_{\ell_1,\ell_2,\ell_3}(n)$ and $c_{\ell_1,\ell_2,\ell_3}(n; g_0) = 0$ for $g_0 > 0$.

Then an immediate induction on g proves that

$$c_{\ell_1,\ell_2,\ell_3}(n) = \sum_{g_0\geq 0} c_{\ell_1,\ell_2,\ell_3}(n; g_0).$$

Note that the parameter g_0 does not *a priori* have a combinatorial interpretation. It is a computational artefact introduced to mimic an induction relation on planar objects that we shall see in the next section.

5.3.2. *A planar object with (almost) the same induction.* Denote by $d_{\ell_1,\ell_2,\ell_3}(n; g_0)$ the number of planar 3-constellations of size $n - g_0$ endowed with:

- a C-permutation of its circle vertices of each color with respectively ℓ_1, ℓ_2 and ℓ_3 cycles;
- An unordered set of g_0 triples of square leaves, such that the triples are mutually disjoint and each triple is of the form f_1, f_2, f_3, with f_i connected to a circle vertex of color i.

Lemma 24. *These numbers satisfy the induction (with g defined as usual by $\ell_1 + \ell_2 + \ell_3 = 2n - 2g + 1$), for $g > 0$:*

$$2g\, d_{\ell_1,\ell_2,\ell_3}(n; g_0) = 2n(n-1)^2 d_{\ell_1,\ell_2,\ell_3}(n-1; g_0-1) + \sum_{h\geq 1} \binom{\ell_1 + 2h}{2h+1} d_{\ell_1+2h,\ell_2,\ell_3}(n; g_0)$$
$$+ \binom{\ell_2 + 2h}{2h+1} d_{\ell_1,\ell_2+2h,\ell_3}(n; g_0) + \binom{\ell_3 + 2h}{2h+1} d_{\ell_1,\ell_2,\ell_3+2h}(n; g_0).$$

Proof. First, we have

$$g_0\, d_{\ell_1,\ell_2,\ell_3}(n; g_0) = n(n-1)^2 d_{\ell_1,\ell_2,\ell_3}(n-1; g_0-1).$$

Indeed, the left-hand side counts the same objects as above with a marked triple of square leaves. If we erase this triple, we get objects counted by $d_{\ell_1,\ell_2,\ell_3}(n-1; g_0-1)$. This can be inverted if we remember in which corners the leaves were attached ($n(n-1)^2$ possibilities ; beware of the root).

Second, the induction for C-permutations leads to

$$2(g-g_0) d_{\ell_1,\ell_2,\ell_3}(n; g_0) = \sum_{h\geq 1} \binom{\ell_1 + 2h}{2h+1} d_{\ell_1+2h,\ell_2,\ell_3}(n; g_0)$$
$$+ \binom{\ell_2 + 2h}{2h+1} d_{\ell_1,\ell_2+2h,\ell_3}(n; g_0) + \binom{\ell_3 + 2h}{2h+1} d_{\ell_1,\ell_2,\ell_3+2h}(n; g_0).$$

□

Corollary 25. *For each $n, g_0, \ell_1, \ell_2, \ell_3$, we have:*

$$2^{2n+1}(n-g_0)\, c_{\ell_1,\ell_2,\ell_3}(n; g_0) = n\, d_{\ell_1,\ell_2,\ell_3}(n; g_0).$$

Proof. Denote by $u_{\ell_1,\ell_2,\ell_3}(n; g_0)$ the left-hand side and by $v_{\ell_1,\ell_2,\ell_3}(n; g_0)$ the right-hand side. For $g = 0$ and $g_0 = 0$ the left-hand side equals the right-hand side (the factor 2^{2n+1} is due to the signs of the cycles —a cycle of length 1 on each of the $2n+1$ vertices— for the objects on the right-hand side). For $g = 0$ and $g_0 > 0$ both sides are zero. For $g > 0$, we have the inductive relation (inherited from the inductive relation for $c_{\ell_1,\ell_2,\ell_3}(n; g_0)$)

$$2g u_{\ell_1,\ell_2,\ell_3}(n; g_0) = 2n^2(n-1) u_{\ell_1,\ell_2,\ell_3}(n-1; g_0-1) + \sum_{h\geq 1} \binom{\ell_1 + 2h}{2h+1} u_{\ell_1+2h,\ell_2,\ell_3}(n; g_0)$$
$$+ \binom{\ell_2 + 2h}{2h+1} u_{\ell_1,\ell_2+2h,\ell_3}(n; g_0) + \binom{\ell_3 + 2h}{2h+1} u_{\ell_1,\ell_2,\ell_3+2h}(n; g_0),$$

and we can see that $v_{\ell_1,\ell_2,\ell_3}(n; g_0)$ satisfies exactly the same relation (inherited from the inductive relation for $d_{\ell_1,\ell_2,\ell_3}(n; g_0)$). Hence the left-hand side equals the right-hand side for all values of $\ell_1, \ell_2, \ell_3, n, g_0$. □

5.3.3. *Final computation.* The nice feature in this statement is that, as it counts planar objects, $d_{\ell_1,\ell_2,\ell_3}(n; g_0)$ can be easily computed combinatorially. As usual, denote $g = 1/2(2n + 1 - \ell_1 - \ell_2 - \ell_3)$.

Lemma 26. *For any integers $\ell_1, \ell_2, \ell_3 \geq 1$ and $g_0 \geq 0$, one has*

$$d_{\ell_1,\ell_2,\ell_3}(n; g_0) = \frac{2^{\ell_1+\ell_2+\ell_3}\,(n!)^3\,(n-g_0)}{n^2\, g_0!\, \ell_1!\, \ell_2!\, \ell_3!} \cdot \sum_{g_1+g_2+g_3=g-g_0} \prod_{i=1}^{3} \frac{P_{g_i}(\ell_i)}{(n-\ell_i-g_0-2g_i)!},$$

where

$$P_h(x) = \sum_{\gamma \vdash h} \frac{(x)_{\ell(\gamma)}}{\prod_i m_i(\gamma)!(2i+1)^{m_i(\gamma)}}.$$

Proof. Fix $g_1, g_2, g_3 \geq 0$ with $g_1+g_2+g_3 = g-g_0$. The number $c_{r_1,r_2,r_3}(n-g_0)$ of planar 3-constellations with $r_1 := \ell_1+2g_1$ (resp. $r_2 := \ell_2+2g_2$ and $r_3 := \ell_3+2g_3$) circle vertices of color 1 (resp. 2 and 3) is given by the multitype (3 colors here) Narayana number

$$\frac{1}{n-g_0}\binom{n-g_0}{r_1}\binom{n-g_0}{r_2}\binom{n-g_0}{r_3}.$$

Using the same arguments as in Section 3.1, we see that the number of possible choices for the C-permutation of the circle vertices of such a constellation with ℓ_1 (resp. ℓ_2 and ℓ_3) cycles of vertices of color 1 (resp. 2 and 3) is

$$\prod_{i=1}^{3} 2^{\ell_i} \frac{(r_i)!}{(\ell_i)!} P_{g_i}(\ell_i).$$

Finally, one has $(n-g_0)\dots(n-1)$ ways to add one by one g_0 square leaves connected to vertices of color 2 (resp. 3). Because of the root, there are $(n-g_0+1)\dots(n)$ to do it for vertices of color 1. As we added these leaves one by one, we can pack them into triples in a canonical way. But these triplets are ordered, so we shall divide by $g_0!$ at the end.

Putting everything together, we get the formula stated in the lemma. □

Finally, from Corollary 25 and Lemma 26, we get a combinatorial proof of the following enumeration formula for 3-constellations with respect to the number of circle vertices of each color.

Proposition 27. *For any $n, \ell_1, \ell_2, \ell_3 \geq 1$, with g defined by $\ell_1+\ell_2+\ell_3 = 2n-2g+1$, one has:*

$$c_{\ell_1,\ell_2,\ell_3}(n) = \frac{n!^2\,(n-1)!}{2^{2g}\;\ell_1!\,\ell_2!\,\ell_3!} \cdot \sum_{g_0+g_1+g_2+g_3=g} \frac{1}{g_0!} \prod_{i=1}^{3} \frac{P_{g_i}(\ell_i)}{(n-\ell_i-g_0-2g_i)!},$$

where

$$P_h(x) = \sum_{\gamma \vdash h} \frac{(x)_{\ell(\gamma)}}{\prod_i m_i(\gamma)!(2i+1)^{m_i(\gamma)}}.$$

This formula can alternatively be deduced from the Poulalhon Schaeffer formula using [18, Lemma 4.1]. If we could refine (15) so as to control the degree distribution, we could give a purely combinatorial proof of the Poulalhon-Schaeffer formula in the case $m = 3$.

5.4. **Conclusion on constellations.** We are unfortunately not able to give a combinatorial proof of the Poulalhon-Schaeffer formula (even in the case $m = 3$). Nevertheless, the work presented here suggests that the different elements in this formula have a combinatorial meaning. The case $m > 3$ seems even harder.

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